PAPER CODE NO.
MATH224

THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2001 EXAMINATIONS

Bachelor of Arts : Year 2
Bachelor of Science : Year 1
Bachelor of Science : Year 2
Bachelor of Science : Year 3
Master of Mathematics : Year 2
Master of Physics : Year 2

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two hours and a half

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B. The marks shown against the questions, or parts of questions, indicate their relative weights. The total of marks available in Section A is 55.

SECTION A

1. Find the solution of

$$\frac{dy}{dx} = \frac{(1+x^2)}{(y+3)}$$

where y(0)=2, putting your answer in the form $y = f(x)$.

[4 marks]

2. Find the general solution of the differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

[4 marks]

3. Solve the system of differential equations

$$\begin{aligned} 5\dot{x} &= 8x - y \\ 5\dot{y} &= 7y - 6x \end{aligned}$$

given the initial conditions $x(0) = 2, y(0) = 1$.

[9 marks]

4. (i) The Laplace transform of the function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

Show that

(i)

$$\mathcal{L}\{e^{-kt} f(t)\} = F(s+k) \quad \text{where } k \text{ is a constant,}$$

[3 marks]

(ii)

$$\mathcal{L}\{t f(t)\} = -\frac{dF(s)}{s}$$

[3 marks]

(iii)

$$\mathcal{L}\{1\} = \frac{1}{s}$$

[2 marks]

Hence show that $\mathcal{L}\{t\} = \frac{1}{s^2}$ and find $\mathcal{L}\{t^3\}$. Hence or otherwise obtain

$$\mathcal{L}^{-1}\{(s+3)^{-4}\}.$$

[5 marks]

5. The function $u(x,t) = F(x)G(t)$ is a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}.$$

Derive an equation for $F(x)$ alone, giving your argument in full.

[5 marks]

.

Show that $F(x) = \sin(2n\pi x/d)$, where n is an integer and d is a constant, satisfies this equation and the boundary conditions

$$u(0,t) = u(d,t) = 0.$$

[4 marks]

.

6. Show that the change of variable

$$\xi = x - y, \quad \eta = y,$$

reduces the partial differential equation

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

to a canonical form.

[5 marks]

Hence find the general solution for $u(x, y)$.

[3 marks]

7. Write down the Cauchy-Riemann equations involving u(x,y) and its conjugate harmonic function $v(x, y)$.
Show that the function

$$u(x, y) = x^3 - 3xy^2 - y^2 + x^2$$

satisfies the two-dimensional Laplace's equation.

[4 marks]

Find $v(x, y)$.

[4 marks]

SECTION B

8. The equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 20\cos(t)$$

has the initial conditions: $y(0) = 2, y'(0) = 6$.

[7 marks]

(i) Find the solution of this problem without using the Laplace transform.

[7 marks]

.

(ii) Laplace transform the equation and find the value of $Y(s)$, the Laplace transform of $y(t)$. Hence find the solution, stating explicitly each inverse Laplace transform you use.

[8 marks]

9. (i) If a is a constant and $F(s)$ is the Laplace transform of $f(t)$, show that

$$\begin{aligned} \mathcal{L}\{e^{at} f(t)\} &= F(s-a) \\ \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} &= e^{-t}\cos(2t)\ . \end{aligned}$$

[4 marks]

(ii) Define the Heaviside function $H(t-a)$. Show how it can be used to describe an impulse of duration 2 units of time, starting at t=0.

[3 marks]

(ii) $x(t)$ and $y(t)$ have initial conditions: $x(0) = 1, y(0) = 2$. They satisfy:

$$\begin{aligned} \frac{dx}{dt} + x - 4y &= 0 \\ x + \frac{dy}{dt} + y &= 0 \end{aligned}$$

Find $x(t)$ and $y(t)$ using the method of Laplace transforms.

[8 marks]

10. Sketch the graph of the function $f(t)$ where:

$$f(t) = 1 - 2\left|\sin\left(\frac{\pi t}{l}\right)\right| \qquad -l \le t \le l\ ;$$

and such that $f(t) = f(t+2l)$ for all x. What is its period?

[5 marks]

Calculate its Fourier series.

[10 marks]

11. A function $u(x, y)$ satisfies Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$ together with the homogeneous boundary conditions

$$u(0, y) = u(a, y) = 0, \quad 0 < y < b$$

on $x = 0$ and $x = a$.

(i) Show that the separable solutions of this boundary value problem are

$$u_n = \sin\left(\frac{n\pi x}{a}\right)\left(C_n \cosh\left(\frac{n\pi y}{a}\right) + D_n \sinh\left(\frac{n\pi y}{a}\right)\right)$$

where n is an integer and C_n and D_n are constants.

[8 marks]

(ii) Find the solution to this problem, i.e. find C_n and D_n, given that $u(x, y)$ satisfies the boundary conditions

$$u(x, 0) = 1, u(x, b) = 0, \quad 0 < x < a$$

on $y = 0$ and $y = b$.

12.

(i) The function $u(x, y)$ satisfies the first order partial differential equation

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = x^2 - y^2,$$

in the domain $x > 0, y > 0$ and the boundary condition $u = x^2$ along $y = 0$. Show that the family of characteristics of this equation are given by

$$x = s\cos(t), y = -s\sin(t)$$

[8 marks]

(ii) Hence determine the function $u(x, y)$.

[7 marks]

12. (i) Writing $\tilde{f}(s)$ for the Laplace transform of $f(t)$ and $H(t-a)$ for the Heaviside (or unit step) function, show that the Laplace transform of $f(t-a)H(t-a)$ is

$$\tilde{f}(s)\exp(-as)\ .$$

[3 marks]

(ii) The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0\ , \qquad 0 < x < 1,\ \ t > 0\ ,$$

and the initial and the boundary conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \frac{\partial u}{\partial x}(x,0) = 0\ ,$$

$$u(0,t) = 0, \quad u(1,t) = t\ .$$

Show that the Laplace transform of $u(x,t)$ with regard to t, denoted by $\tilde{u}$, satisfies the ordinary differential equation

$$\tilde{u}'' - 2s\tilde{u}' + s^2\tilde{u} = 0\ .$$

[5 marks]

(iii) Find the boundary conditions for $\tilde{u}$ at $x = 0$ and at $x = 1$.

[2 marks]

(iv) Solve this equation for $\tilde{u}$ and hence find the function $u(x,t)$.

[5 marks]