PAPER CODE NO.
MATH224

THE UNIVERSITY
of LIVERPOOL

MAY 2007 EXAMINATIONS

Bachelor of Arts	:	Year 2
Bachelor of Science	:	Year 1
Bachelor of Science	:	Year 2
Bachelor of Science	:	Year 3
Master of Chemistry	:	Year 2
Master of Mathematics	:	Year 2
Master of Physics	:	Year 2
Master of Physics	:	Year 4
No qualification aimed for	:	Year 1

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B. The marks shown against the questions, or parts of questions, indicate their relative weights. The total of marks available in Section A is 55.

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SECTION A

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{2y^2}{(x^2+4)}$$

putting your answer in the form $y = f(x)$.

[**5 marks**]

2. Solve the initial value problem

$$\frac{dy}{dx} - y\tan x = 2\cos x - \sec x$$

for $y(x)$ where $y(0) = 2$.

[**6 marks**]

3. Solve the system of differential equations

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = y$$

given the initial conditions $x(0) = -4$ and $y(0) = 3$.

[**8 marks**]

4. Find the general solution to the ordinary differential equation

$$y'' - 2y' + (1-a^2)y = e^{bx}$$

when $a \neq 0$ and $b \neq 1 \pm a$ where a and b are real parameters.
Find the general solution in the case when $b = 1 + a$ and $a \neq 0$.

[**10 marks**]

5. The function $f(x)$ is odd and has period 2π and also satisfies

$$f(x) \; = \; x \qquad \text{for} \; - \; \pi \; \leq \; x \; \leq \pi \; .$$

Sketch the graph of $f(x)$ for $- \; 3\pi < x < 3\pi$ and find its Fourier series.

[**8 marks**]

6. The function $u(x, y)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} \; + \; \frac{1}{x}\frac{\partial u}{\partial y} \; = \; 0$$

in the region $x > 0$.
(i) Find the characteristic curves of this equation for problems with a boundary condition on the line $x = 1$.
(ii) Hence find the solution to the boundary value $u = y^2$ when $x = 1$.

[**9 marks**]

7. Write down the Cauchy-Riemann equations connecting a function $u(x, y)$ to its conjugate harmonic function $v(x, y)$.
Show that the function

$$u(x, y) \; = \; e^{x-y} \cos(x + y)$$

satisfies the two dimensional Laplace's equation.
Find $v(x, y)$ the conjugate harmonic function corresponding to $u(x, y)$.

[**9 marks**]

SECTION B

8. Find the solution of the differential equation

$$x^2y'' - 9xy' + 9y = 2x^3 + x^2$$

with the initial conditions $y(1) = 1$ and $y'(1) = 2$.

[15 marks]

9. Find the general solution of the system of differential equations

$$\frac{dx}{dt} = 2x + y + \cos t$$
$$\frac{dy}{dt} = x + 2y + \sin t \ .$$

[15 marks]

10. The function $u(x, y)$ satisfies the first order partial differential equation

$$x\frac{\partial u}{\partial x} + (x+y)\frac{\partial u}{\partial y} = u + 2x + y$$

in $x > 0$ subject to the boundary conditions $u(x, 0) = x$.
Show that the family of characteristics of the partial differential equation may be represented by

$$x = se^t \qquad y = ste^t$$

where s and t are parameters whose significance you should explain.
Hence determine the function $u(x, y)$.

[15 marks]

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11. (i) Sketch the graph of the function $g(t)$ in the range $-2T \leq t \leq 2T$ where

$$g(t) \;=\; 1 \;+\; \left|\cos\left(\frac{\pi t}{T}\right)\right| \qquad -T \;\leq\; t \;\leq\; T\;.$$

What is the period of $g(t)$?
(ii) Calculate the Fourier series for $g(t)$.
[You may use without proof the relation

$$\cos A \cos B \;=\; \tfrac{1}{2}\left(\cos(A+B) \;+\; \cos(A-B)\right)\;.]$$

[15 marks]

12. Show that the function

$$u(x,t) \;=\; A \;+\; Bx \;+\; \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2}$$

where A, B and C_n are constants, satisfies the heat equation

$$\frac{\partial u}{\partial t} \;=\; k\frac{\partial^2 u}{\partial x^2}$$

in a horizontal bar of length L where the left and right ends of the bar are held at temperatures T_0 and T_1 respectively and k is a positive constant.
If the initial temperature distribution is

$$u(x,0) \;=\; \begin{cases} 0^{\mathrm{o}}\mathrm{C} & 0 \;<\; x \;<\; \frac{1}{2}L \\ 100^{\mathrm{o}}\mathrm{C} & \frac{1}{2}L \;<\; x \;<\; L \end{cases}$$

and the ends are held at 20^{o}C, find the temperature distribution, $u(x,t)$, at time t.

[15 marks]