PAPER CODE NO.
MATH224

THE UNIVERSITY *of* LIVERPOOL

MAY 2006 EXAMINATIONS

Bachelor of Arts	:	Year 2
Bachelor of Science	:	Year 1
Bachelor of Science	:	Year 2
Bachelor of Science	:	Year 3
Master of Chemistry	:	Year 2
Master of Mathematics	:	Year 2
Master of Physics	:	Year 2
Master of Physics	:	Year 4
No qualification aimed for	:	Year 1

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B. The marks shown against the questions, or parts of questions, indicate their relative weights. The total of marks available in Section A is 55.

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SECTION A

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{3xy}{(x^2 - 1)}$$

putting your answer in the form $y = f(x)$.

[5 marks]

2. Solve the initial value problem

$$x^3 \frac{dy}{dx} - xy = 2e^{-\frac{1}{x}}$$

for $y(x)$ where $y(1) = 0$.

[6 marks]

3. Solve the system of differential equations

$$\frac{dx}{dt} = 3x + 2y$$
$$\frac{dy}{dt} = x + 2y$$

given the initial conditions $x(0) = -1$ and $y(0) = 4$.

[8 marks]

4. The function $f(x)$ is even and has period 2π and also satisfies

$$f(x) = \begin{cases} 2 & 0 < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \, . \end{cases}$$

Sketch the graph of $f(x)$ for $-2\pi < x < 2\pi$ and find its Fourier series.

[8 marks]

5. The function $u(x,y)$ satisfies the partial differential equation

$$x^2y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

in the region $x > 0$.
(i) Find the characteristic curves of this equation for problems with a boundary condition on the line $y = 0$.
(ii) Hence find the solution to the boundary value $u = \frac{1}{x}$ when $y = 0$ and $x > 0$.

[9 marks]

6. The function $u(x,t) = F(x)\sin(\lambda ct)$, where c and λ are positive constants, is a non-trivial solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} .$$

Show that $F(x)$ satisfies the ordinary differential equation

$$F''(x) + \lambda^2 F(x) = 0 .$$

Given that $u(x,t)$ also satisfies the boundary conditions $u(x,0) = 0$ and

$$u(0,t) = u(L,t) = 0$$

for all t, show that the possible values of λ are $n\pi/L$ where n is an integer and find the corresponding functions of $F(x)$.
Sketch $F(x)$ on the interval $0 \le x \le L$ for $n = 5$.
Write down the general solution for $u(x,t)$.

[10 marks]

7. Write down the Cauchy-Riemann equations connecting a function $u(x,y)$ to its conjugate harmonic function $v(x,y)$.
Show that the function

$$u(x,y) = \frac{x}{(x^2+y^2)}$$

satisfies the two dimensional Laplace's equation if $x^2 + y^2 \neq 0$.
Find $v(x,y)$ the conjugate harmonic function corresponding to $u(x,y)$.

[9 marks]

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S E C T I O N   B

8. Find the solution of the differential equation

$$x^2y'' \;-\; 5xy' \;+\; 5y \;=\; 4x^4 \;+\; 2x^2$$

with the initial conditions $y(1) = 4$ and $y'(1) = \frac{1}{3}$.

[15 marks]

9. Find the general solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 3x \;-\; y \;+\; 2e^{2t} \\ \frac{dy}{dt} &= -\,6x \;+\; 4y \;-\; e^{2t}\;. \end{aligned}$$

[15 marks]

10. The function $u(x,y)$ satisfies the first order partial differential equation

$$(1+2y)\frac{\partial u}{\partial x} \;+\; y\frac{\partial u}{\partial y} \;=\; u \;+\; 2y$$

in the domain $x > 0$, $y > 0$ and the boundary conditions $u = 2y(1-2y)$ on $x = 0$.
Show that the family of characteristics of the partial differential equation may be represented by

$$x \;=\; t \;+\; 2s(e^t - 1) \qquad y \;=\; se^t$$

where s and t are parameters whose significance you should explain.
Hence determine the function $u(x,y)$.

[15 marks]

11. (i) Sketch the graph of the function $g(t)$ in the range $-2\pi \leq t \leq 2\pi$ where

$$g(t) \;=\; |2\sin(2t)|\ .$$

What is the period of $g(t)$?
(ii) Calculate the Fourier series for $g(t)$.
[You may use, without proof, the result

$$\sin A \cos B \;=\; \frac{1}{2}\sin(A+B) \;+\; \frac{1}{2}\sin(A-B)\ .\Big]$$

[15 marks]

12. Show that the function

$$u(x,y) \;=\; \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n\cosh\left(\frac{n\pi y}{L}\right) \;+\; B_n\sinh\left(\frac{n\pi y}{L}\right)\right]$$

satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} \;+\; \frac{\partial^2 u}{\partial y^2} \;=\; 0$$

in the square $0 \leq x,\, y \leq L$ with boundary conditions $u(0,y) = u(L,y) = 0$.
Find the particular solution for which $u(x,0) = 0$ and $u(x,L) = x(L-x)$.

[15 marks]