PAPER CODE NO.
MATH224

THE UNIVERSITY
of LIVERPOOL

SUMMER 2005 EXAMINATIONS

Bachelor of Arts : Year 2
Bachelor of Science : Year 1
Bachelor of Science : Year 2
Bachelor of Science : Year 3
Master of Mathematics : Year 2
Master of Physics : Year 2
No qualification aimed for : Year 1

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B. The marks shown against the questions, or parts of questions, indicate their relative weights. The total of marks available in Section A is 55.

THE UNIVERSITY
of LIVERPOOL

SECTION A

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = y^4(x + \cos x),$$

putting your answer in the form $y = f(x)$.

[5 marks]

2. Solve the initial value problem

$$xy^2 \frac{dy}{dx} = x^3 + y^3 \; ; \quad y(1) = 2 \; .$$

[6 marks]

3. Solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 4x + 5y \\ \frac{dy}{dt} &= 2x + y \end{aligned}$$

given the initial conditions $x(0) = 1, y(0) = 1$.

[9 marks]

4. The function $f(t)$ has period 2π and is odd. It also satisfies

$$f(t) = \begin{cases} 0\,, & 0 < t < \frac{\pi}{2} \\ \\ 2\,, & \frac{\pi}{2} < t < \pi\,. \end{cases}$$

Sketch the graph of the function for $-2\pi < t < 2\pi$ and find its Fourier series.

Hint: There is no need to simplify trigonometrical expressions such as $\cos\frac{n\pi}{2}$.

[10 marks]

5. Find the solution to the differential equation

$$\frac{dz}{dt} + 3z\cos t = \cos t$$

with the initial condition $z(0) = 3$.

[7 marks]

6. (i) Find the general solution to the differential equation

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 3\frac{dy}{dt} = e^{4t} .$$

[7 marks]

(ii) How many initial conditions would be needed to determine a unique solution?

[2 marks]

7. (i) Write down the Cauchy-Riemann equations connecting a function $u(x, y)$ to its conjugate harmonic function $v(x, y)$.
Show that the function

$$u(x, y) = 3\cosh(4x)\cos(4y)$$

satisfies the two-dimensional Laplace's equation.

[4 marks]

(ii) Find $v(x, y)$, the conjugate harmonic function corresponding to $u(x, y)$ in part (i).

[5 marks]

SECTION B

8. Find the solution of the differential equation

$$x^2y'' + xy' - y = 2 + \sqrt{x}$$

with the initial conditions $y(1) = 3, y'(1) = 1$.

[15 marks]

9. Find the general solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x + y + 1 + t\ , \\ \frac{dy}{dt} &= 4x + y + 2t\ . \end{aligned}$$

[15 marks]

10. The function $f(x)$ is even and has period L. Its value in the region $0 \leq x < L$ is

$$f(x) = x(L - x) \ , \qquad 0 \leq x < L$$

 (i) Sketch the graph of the function for $-2L < x < 2L$.

[4 marks]

 (ii) Find the Fourier series of $f(x)$.

[11 marks]

11. The function $u(x, y)$ satisfies the first order partial differential equation

$$\frac{\partial u}{\partial x} - e^x \frac{\partial u}{\partial y} = 3u$$

in the domain $y > 0$. On the boundary, $y = 0$, the value of u is given by

$$u(x, 0) = x \ .$$

 (i) Show that the family of characteristics of this partial differential equation may be represented by

$$x = s + t \ , \qquad y = e^s - e^{s+t}$$

where s and t are parameters whose significance you should explain.

[7 marks]

 (ii) Hence, or otherwise, determine the function $u(x, y)$.

[8 marks]

12. The temperature $U(\theta, t)$ in a metal ring obeys the heat equation

$$\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial \theta^2} ,$$

where the angular coordinate θ will be chosen to run from $-\pi$ to π.

(i) By considering the separable solutions of the heat equation show that the general solution to the heat equation in the ring can be written as

$$U(\theta, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\kappa n^2 t} \left[a_n \cos n\theta + b_n \sin n\theta \right]$$

where n is an integer and a_n and b_n are constants.

[7 marks]

(ii) Initially the temperature is 100 °C for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and 0 °C in the rest of the ring. Find the temperature distribution at later times.

[8 marks]