PAPER CODE NO.
MATH224

THE UNIVERSITY
of LIVERPOOL

SUMMER 2004 SOLUTIONS EXAMINATIONS

Bachelor of Arts : Year 2
Bachelor of Science : Year 1
Bachelor of Science : Year 2
Bachelor of Science : Year 3
Master of Mathematics : Year 2
Master of Physics : Year 2
Master of Physics : Year 4
No qualification aimed for : Year 1

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two hours and a half

INSTRUCTIONS TO CANDIDATES

These are brief solutions, so that you can check your answers. You will need to show more working than you see here.

SECTION A

1. *Find the general solution of the differential equation*

$$\frac{dy}{dx} = e^y(x^2+2),$$

putting your answer in the form $y = f(x)$.

[5 marks]

This is a separable equation. Separating x from y gives the integrals

$$\int e^{-y} dy = \int (x^2+2) dx$$

and the final result is

$$y = -\ln\left(A - \tfrac{1}{3}x^3 - 2x\right)$$

2. *Solve the initial value problem*

$$\frac{dy}{dx} = 4\frac{y}{x} - x^2\ ; \quad y(1) = 0\ .$$

[5 marks]

This is a linear equation. You can simplify it by using the integrating factor

$$\mu = \exp\left[-\int \frac{4}{x} dx\right] = e^{-4\ln x} = x^{-4}$$

Using the integrating factor (or otherwise) gives the general solution of the differential equation:

$$y = x^3 + Cx^4$$

The boundary condition $y(1) = 0$ implies $C = -1$ so the final answer is

$$y = x^3 - x^4$$

3. *Solve the system of differential equations*

$$\begin{aligned}\frac{dx}{dt} &= 2x - 2y \\ \frac{dy}{dt} &= -x + 3y\end{aligned}$$

given the initial conditions $x(0) = 1, y(0) = 1$.

[9 marks]

The two main ways of doing this are by the matrix method (using eigenvalues and eigenvectors) or by elimination. I'll show the matrix method here (see Q9 for elimination). As a matrix the problem reads

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

We find the eigenvalues from the equation

$$\begin{vmatrix} 2-\lambda & -2 \\ -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \cdots \Rightarrow (\lambda - 4)(\lambda - 1) = 0$$

so the two eigenvalues are $\lambda = 4$ and $\lambda = 1$. To find the eigenvalue for $\lambda = 4$ we solve

$$\begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

giving the eigenvector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{for} \quad \lambda = 4.$$

Similarly the other eigenvector is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{for} \quad \lambda = 1.$$

The general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + B\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t}$$

To match the initial conditions we need $A = -\frac{1}{3}$ and $B = \frac{2}{3}$, so the final solution is

$$\begin{aligned} x &= -\tfrac{1}{3}e^{4t} + \tfrac{4}{3}e^{t} \\ y &= \tfrac{1}{3}e^{4t} + \tfrac{2}{3}e^{t} \end{aligned}$$

(Elimination gives the same result.)

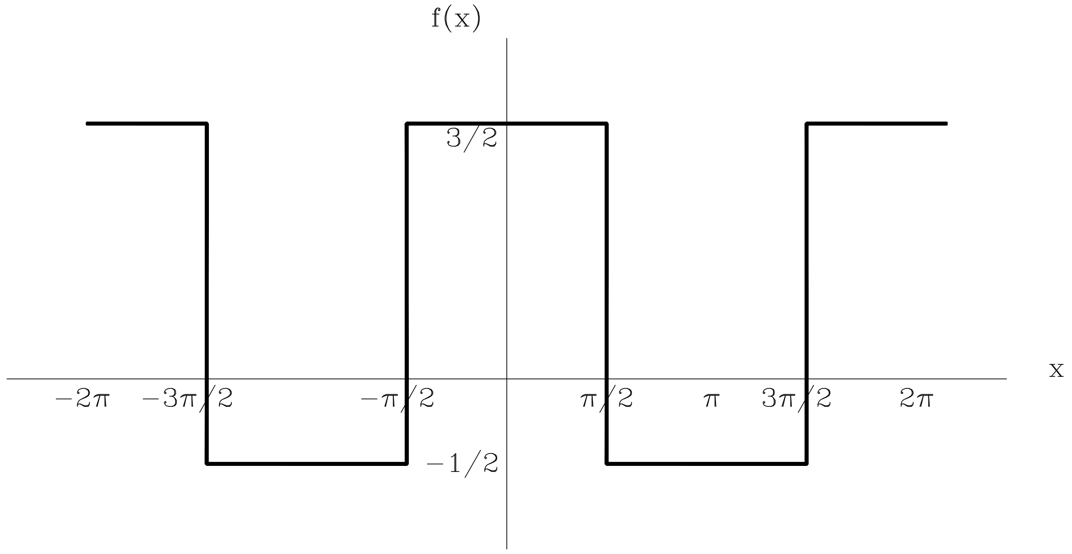
4. *The function $f(x)$ is even and has period 2π; it also satisfies*

$$f(x) = \begin{cases} \frac{3}{2}\,, & 0 < x < \pi/2 \\ -\frac{1}{2}\,, & \pi/2 < x < \pi\,. \end{cases}$$

Sketch the graph of the function for $-2\pi < x < 2\pi$ and find its Fourier series.

[10 marks]

The question tells us the function's value between $x = 0$ and π. Using periodicity and the fact that the function is even we can find f for any other x, the sketch should look like



Because the function is even we know immediately that $b_n = 0$ for all n. To find the a_n coefficients we have to do the integrals

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} f(x)\ \cos nx\ dx$$

First when $n = 0$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\ dx \\ &= \frac{1}{\pi}\int_0^{\pi/2} \frac{3}{2}\ dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}\left(-\frac{1}{2}\right)\ dx + \frac{1}{\pi}\int_{3\pi/2}^{2\pi}\frac{3}{2}\ dx \\ a_0 &= \frac{3}{4} - \frac{1}{2} + \frac{3}{4} = 1 \end{aligned}$$

Next $n \neq 0$.

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\ \cos nx\ dx = \frac{2}{\pi}\int_0^{\pi} f(x)\ \cos nx\ dx \qquad \text{(by symmetry)} \\ &= \frac{2}{\pi}\int_0^{\pi/2}\frac{3}{2}\cos nx\ dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}\left(-\frac{1}{2}\right)\cos nx \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{\pi}\left[\frac{1}{n}\sin nx\right]_0^{\pi/2} - \frac{1}{\pi}\left[\frac{1}{n}\sin nx\right]_{\pi/2}^{\pi} \\
a_n &= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

Putting this together

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\
&= \frac{1}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos nx \\
&= \frac{1}{2} + \frac{4}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \cdots\right] \\
&= \frac{1}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}(-1)^k\cos(2k+1)x
\end{aligned}
$$

(any of these three answers is fine.)

5. *The function $u(x, y)$ satisfies the partial differential equation*

$$3xy\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \; .$$

(i) Find the characteristic curves of this equation for problems with a boundary condition on the line $y = 0$.

[5 marks]

(ii) Hence find the solution to the boundary value problem

$$u = \cos x \quad \text{when} \quad y = 0 \; .$$

[4 marks]

(i) The parametric equations for the characteristic curves are

$$\frac{dx}{dt} = 3xy, \qquad\qquad \frac{dy}{dt} = 1$$

Since the boundary is on the line $y = 0$ a good initial condition is

$$y = 0; \qquad x = s \qquad \text{when } t = 0.$$

Solve the y equation first

$$\frac{dy}{dt} = 1 \quad \text{with} \quad \mathrm{y(0) = 0} \qquad \Rightarrow \mathrm{y = t}$$

Use this to eliminate y from the equation for x

$$\frac{dx}{dt} = 3xt$$

This is a separable equation, its solution (with $x(0) = s$) is

$$x = s\exp\left(\tfrac{3}{2}t^2\right)$$

The characteristic curves are

$$x(t) = s\exp\left(\tfrac{3}{2}t^2\right), \qquad y(t) = t$$

(ii) There rhs of this PDE is 0, so the equation for u is simply

$$\frac{du}{dt} = 0 \qquad \text{with initial condition} \quad u(0) = \cos s$$

and the solution is just

$$u(t) = \cos s \quad \text{for all } t.$$

We want a solution with x and y, so we eliminate s and t by using

$$t = y, \qquad s = x\exp\left(-\tfrac{3}{2}t^2\right) = x\exp\left(-\tfrac{3}{2}y^2\right)$$

to give the final answer

$$u(x, y) = \cos\left[xe^{-\frac{3}{2}y^2}\right]$$

6. (i) *Given that* $u(x,y) = F(x)\exp(-\mu y)$*, where* μ *is a constant, satisfies the partial differential equation*

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

show that the function F *must obey*

$$\frac{d^2 F}{dx^2} + \mu^2 F = 0.$$

Write down the general solution for F

[4 marks]

(ii) If the function u *also satisfies the boundary conditions*

$$u(0,y) = 0, \qquad u(L,y) = 0$$

where L *is a constant, find the possible values of* μ*. Hence find the general solution for* u *with these boundary conditions.*

[5 marks]

Substitute the suggested form into the equation, and you get

$$e^{-\mu y}\left(\frac{d^2F}{dx^2} + \mu^2 F\right) = 0 \quad \Rightarrow \frac{d^2F}{dx^2} + \mu^2 F = 0$$

The general solution for F is

$$F(x) = A\cos\mu x + B\sin\mu x$$

(ii) To satisfy $u(0,y) = 0$ we need $A = 0$ (no cosine term). To also satisfy $u(L,y)$ we need

$$\sin\mu L = 0 \qquad \mu = \frac{n\pi}{L}$$

with n an integer.

We get the general solution by adding together all the separable solutions we have just found, i.e.

$$u(x,y) = \sum_n C_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n\pi y}{L}\right)$$

7. *(i) Write down the Cauchy-Riemann equations connecting a function* $u(x,y)$ *to its conjugate harmonic function* $v(x,y)$.
Show that the function

$$u(x,y) = 3e^{2x}\cos(2y) - e^y \sin x$$

satisfies the two-dimensional Laplace's equation.

[4 marks]

(ii) Find $v(x,y)$, *the conjugate harmonic function corresponding to* $u(x,y)$ *in part (i).*

[4 marks]

The Cauchy-Riemann equations say

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

To check Laplace, take the second derivatives of u

$$\begin{aligned} u_{xx} &= 12e^{2x}\cos 2y + e^y \sin x \\ u_{yy} &= -12e^{2x}\cos 2y - e^y \sin x \end{aligned}$$

so $u_{xx} + u_{yy} = 0$.
(ii) From the CR equation

$$\frac{\partial v}{\partial y} = u_x = 6e^{2x}\cos 2y - e^y \cos x$$

Integrate both sides wrt y.

$$v = 3e^{2x}\sin 2y - e^y \cos x + A(x)$$

To fix the unknown function A use the other CR relation

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -6e^{2x}\sin 2y - e^y\cos x = -6e^{2x}\sin 2y - e^y \cos x - A'(x) \Rightarrow A'(x) = 0$$

$A(x)$ has to be a constant so the final result is

$$v = 3e^{2x}\sin 2y - e^y \cos x + C$$

SECTION B

8. *Find the solution of the differential equation*

$$x^2 z'' - 3xz' + 3z = x^2 + 1$$

with the initial conditions $z(1) = 0, z'(1) = 1$.

[15 marks]

Recognise that this is Euler's equation, so try functions of the type x^p for the complementary function. The characteristic polynomial equation is

$$p^2 - 4p + 3 = 0$$

which has the roots $p = 1$ or 3. Therefore the complementary function is

$$z_c = Ax^3 + Bx$$

For a particular solution of Euler's equation try a polynomial with the same powers that occur on the right-hand side.

$$z_p = \alpha x^2 + \beta$$

This works if $\alpha = -1, \quad \beta = \frac{1}{3}$, so

$$z_p = -x^2 + \tfrac{1}{3}$$

The general solution is

$$z = z_c + z_p = Ax^3 - x^2 + Bx + \tfrac{1}{3}\ .$$

Now use the initial conditions to find A and B. (A common mistake is to use the initial conditions too early, on just the complementary function. You must impose the initial conditions *after* adding z_c and z_p.) The final answer should be

$$z = \tfrac{7}{6}x^3 - x^2 - \tfrac{1}{2}x + \tfrac{1}{3}$$

9. *Find the general solution of the system of differential equations*

$$\frac{dx}{dt} = 4x - y + e^{3t} ,$$

$$\frac{dy}{dt} = -4x + 4y + e^{3t} .$$

[15 marks]

You can do this either by vector methods or by elimination. Here is the solution by elimination (see Q3 for a matrix example). From the first equation we have

$$y = 4x - \frac{dx}{dt} + e^{3t}$$

Substitute into the second equation to get rid of y.

$$\frac{d^2x}{dt^2} - 8\frac{dx}{dt} + 12x = -2e^{3t}$$

For the complementary function try $e^{\lambda t}$. The solutions for λ are

$$\lambda^2 - 8\lambda + 12 = 0 \qquad \Rightarrow \lambda = 6 \text{or } 2$$

$$x_c = Ae^{6t} + Be^{2t}$$

For the particular solution guess $x_p = \alpha e^{3t}$. This works if $\alpha = \frac{2}{3}$,

$$x_p = \tfrac{2}{3}e^{3t}$$

$$\underline{x = x_c + x_p = Ae^{6t} + Be^{2t} + \tfrac{2}{3}e^{3t}}$$

We recover y from $y = 4x - \frac{dx}{dt} + e^{3t}$ to give

$$\underline{y = -2Ae^{6t} + 2Be^{2t} + \tfrac{5}{3}e^{3t}}$$

10. *The function $u(x, y)$ satisfies the first order partial differential equation*

$$x\frac{\partial u}{\partial x} + (1+x)\frac{\partial u}{\partial y} = 2u - 2y - 3$$

in the domain $x > 0,\ y > 0$ and the boundary condition

$$u(x, 0) = x + 2 \qquad \text{on} \qquad y = 0\ .$$

(i) Show that the family of characteristics of this partial differential equation may be represented by

$$x = se^t\ , \qquad y = se^t + t - s$$

where s and t are parameters whose significance you should explain.

[6 marks]

(ii) Hence, or otherwise, determine the function $u(x, y)$.

[9 marks]

(i) The characteristics are given by the equations

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = 1 + x$$

with the initial condition $y = 0, x = s$ when $t = 0$. Solve the x equation first, because it doesn't mix up x and y.

$$\frac{dx}{dt} = x \qquad \Rightarrow x = se^t$$

Now we can eliminate x and solve the y equation:

$$\frac{dy}{dt} = 1 + se^t \qquad \Rightarrow y = t + se^t + C$$

We want $y = 0$ when $t = 0$, so

$$y = t + se^t - s$$

as we were told.

Explain s and t. s identifies the different characteristic curves, it is constant on any given characteristic. t is a parameter indicating the distance along a characteristic. For convenience we make $t = 0$ on the boundary.

(ii) The parametric equation for u is

$$\frac{du}{dt} = 2u - 2y - 3$$

First step is to get rid of any x or y by using the results from (i).

$$\frac{du}{dt} = 2u - 2se^t - 2t + 2s - 3$$

This is a linear equation, solve it to get

$$u = Ae^{2t} + 2se^t + t + 2 - s$$

with A an unknown constant. At $t = 0$ we were told $u(0) = x(0) + 2 = s + 2$ which tells us that $A = 0$. So

$$u = 2se^t + t + 2 - s$$

Using the results from (i) to eliminate s and t gives

$$u = x + y + 2$$

11. *(i) Sketch the graph of the function $g(t)$ where:*

$$g(t) = 4 + |\sin t|$$

What is its period?

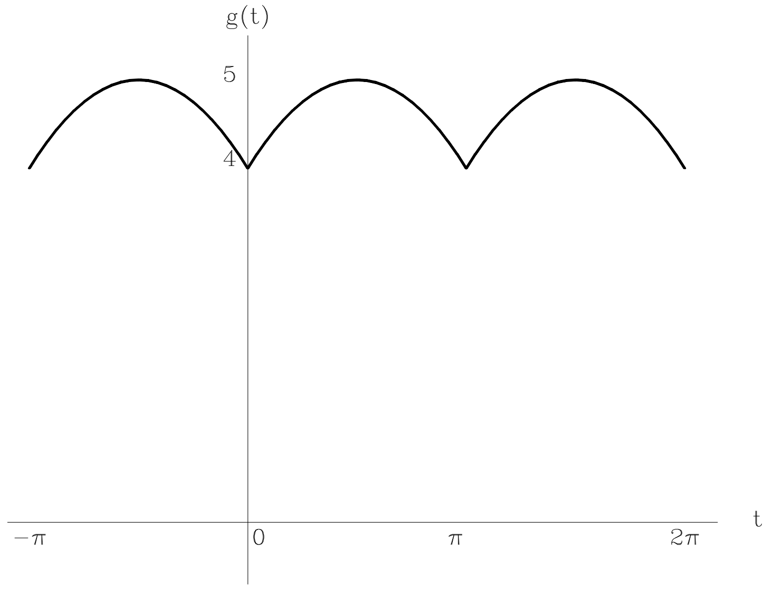
[5 marks]

(ii) Calculate the Fourier series for $g(t)$.

[10 marks]

Hint: *Remember that* $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$ *for any A and B.*

(i) Your sketch should look like this:



The function g has period π.

(ii) Because the function is even with period π its Fourier series will have the form

$$g(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2nt$$

We find the a_n form the integrals

$$a_n = \frac{2}{\pi}\int_0^{\pi} g(t)\cos 2nt\; dt$$

Because $\sin t \geq 0$ in the region $0 \leq t \leq \pi$ we can replace $|\sin t|$ by $\sin t$ *in this region* . The results are

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(4+\sin t)dt = \cdots = 8 + \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(4+\sin t)\cos 2nt\; dt = \cdots = -\frac{4}{\pi}\,\frac{1}{4n^2-1}$$

(using the hint). So

$$g(t) = 4 + \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nt}{4n^2-1}$$

12. *A function $u(x,y)$ satisfies Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$ together with the homogeneous boundary conditions*

$$u(0,y) = u(a,y) = 0, \quad 0 < y < b$$

on $x = 0$ and $x = a$.

(i) Verify that the function

$$u(x,y) = \sum_n \sin\left(\frac{n\pi x}{a}\right)\left[C_n \cosh\left(\frac{n\pi y}{a}\right) + D_n \sinh\left(\frac{n\pi y}{a}\right)\right]$$

satisfies the differential equation and the boundary conditions on $x = 0$ and $x = a$ if n is an integer and C_n and D_n are constants.

[5 marks]

(ii) Find the solution to this problem, i.e. find C_n and D_n, given that $u(x,y)$ satisfies the boundary conditions

$$u(x,0) = 0, \; u(x,b) = 1, \quad 0 < x < a$$

on $y = 0$ and $y = b$.

[10 marks]

(i) Putting in $x = 0$ and $x = a$ gives zero, so the boundary conditions are satisfied. Take the second derivatives

$$\begin{aligned} u_{xx} &= -\sum_n \left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\left[C_n \cosh\left(\frac{n\pi y}{a}\right) + D_n \sinh\left(\frac{n\pi y}{a}\right)\right] \\ u_{yy} &= +\sum_n \left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\left[C_n \cosh\left(\frac{n\pi y}{a}\right) + D_n \sinh\left(\frac{n\pi y}{a}\right)\right] \end{aligned}$$

so the Laplace equation $u_{xx} + u_{yy} = 0$ is fulfilled.

(ii) We can satisfy the b.c. $u(x,0) = 0$ by making $C_n = 0$ for all n.

To find the value of all the D_n we have to calculate the half-range Fourier sine series for the boundary condition $u(x,b) = 1$. The result is

$$\sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi x}{a}\right) = 1, \quad 0 < x < a$$

where $2k+1 = n$. To match this series when $y = b$ needs

$$D_{2k+1} \sinh\left(\frac{(2k+1)\pi b}{a}\right) = \frac{4}{(2k+1)\pi}$$

$$\Rightarrow u(x,y) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \frac{\sin\left(\frac{(2k+1)\pi x}{a}\right) \sinh\left(\frac{(2k+1)\pi y}{a}\right)}{\sinh\left(\frac{(2k+1)\pi b}{a}\right)}$$