PAPER CODE NO.
MATH224

THE UNIVERSITY
of LIVERPOOL

SUMMER 2004 EXAMINATIONS

Bachelor of Arts : Year 2
Bachelor of Science : Year 1
Bachelor of Science : Year 2
Bachelor of Science : Year 3
Master of Mathematics : Year 2
Master of Physics : Year 2
Master of Physics : Year 4

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two hours and a half

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B. The marks shown against the questions, or parts of questions, indicate their relative weights. The total of marks available in Section A is 55.

SECTION A

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = e^y(x^2+2),$$

putting your answer in the form $y = f(x)$.

[5 marks]

2. Solve the initial value problem

$$\frac{dy}{dx} = 4\frac{y}{x} - x^2 \ ; \quad y(1) = 0 \ .$$

[5 marks]

3. Solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 2x - 2y \\ \frac{dy}{dt} &= -x + 3y \end{aligned}$$

given the initial conditions $x(0) = 1, y(0) = 1$.

[9 marks]

4. The function $f(x)$ is even and has period 2π; it also satisfies

$$f(x) = \begin{cases} \frac{3}{2}\,, & 0 < x < \pi/2 \\ -\frac{1}{2}\,, & \pi/2 < x < \pi \ . \end{cases}$$

Sketch the graph of the function for $-2\pi < x < 2\pi$ and find its Fourier series.

[10 marks]

5. The function $u(x,y)$ satisfies the partial differential equation

$$3xy\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0\ .$$

(i) Find the characteristic curves of this equation for problems with a boundary condition on the line $y = 0$.

[5 marks]

(ii) Hence find the solution to the boundary value problem

$$u = \cos x \quad \text{when} \quad y = 0\ .$$

[4 marks]

6. (i) Given that $u(x,y) = F(x)\exp(-\mu y)$, where μ is a constant, satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

show that the function F must obey

$$\frac{d^2 F}{dx^2} + \mu^2 F = 0.$$

Write down the general solution for F

[4 marks]

(ii) If the function u also satisfies the boundary conditions

$$u(0,y) = 0, \qquad u(L,y) = 0$$

where L is a constant, find the possible values of μ. Hence find the general solution for u with these boundary conditions.

[5 marks]

7. (i) Write down the Cauchy-Riemann equations connecting a function $u(x,y)$ to its conjugate harmonic function $v(x,y)$.
Show that the function

$$u(x,y) = 3e^{2x}\cos(2y) - e^{y}\sin x$$

satisfies the two-dimensional Laplace's equation.

[4 marks]

(ii) Find $v(x,y)$, the conjugate harmonic function corresponding to $u(x,y)$ in part (i).

[4 marks]

SECTION B

8. Find the solution of the differential equation

$$x^2 z'' - 3xz' + 3z = x^2 + 1$$

with the initial conditions $z(1) = 0, z'(1) = 1$.

[15 marks]

9. Find the general solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 4x - y + e^{3t} , \\ \frac{dy}{dt} &= -4x + 4y + e^{3t} . \end{aligned}$$

[15 marks]

10. The function $u(x, y)$ satisfies the first order partial differential equation

$$x\frac{\partial u}{\partial x} + (1 + x)\frac{\partial u}{\partial y} = 2u - 2y - 3$$

in the domain $x > 0,\ y > 0$ and the boundary condition

$$u(x, 0) = x + 2 \quad \text{on} \quad y = 0 .$$

(i) Show that the family of characteristics of this partial differential equation may be represented by

$$x = se^t , \qquad y = se^t + t - s$$

where s and t are parameters whose significance you should explain.

[6 marks]

(ii) Hence, or otherwise, determine the function $u(x, y)$.

[9 marks]

11. (i) Sketch the graph of the function $g(t)$ where:

$$g(t) = 4 + |\sin t|$$

What is its period?

[5 marks]

(ii) Calculate the Fourier series for $g(t)$.

[10 marks]

Hint: Remember that $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$ for any A and B.

12. A function $u(x, y)$ satisfies Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$ together with the homogeneous boundary conditions

$$u(0, y) = u(a, y) = 0, \quad 0 < y < b$$

on $x = 0$ and $x = a$.

(i) Verify that the function

$$u(x, y) = \sum_n \sin\left(\frac{n\pi x}{a}\right)\left[C_n \cosh\left(\frac{n\pi y}{a}\right) + D_n \sinh\left(\frac{n\pi y}{a}\right)\right]$$

satisfies the differential equation and the boundary conditions on $x = 0$ and $x = a$ if n is an integer and C_n and D_n are constants.

[5 marks]

(ii) Find the solution to this problem, i.e. find C_n and D_n, given that $u(x, y)$ satisfies the boundary conditions

$$u(x, 0) = 0, \; u(x, b) = 1, \quad 0 < x < a$$

on $y = 0$ and $y = b$.

[10 marks]