PAPER CODE NO.
MATH224

THE UNIVERSITY
of LIVERPOOL

SUMMER 2003 EXAMINATIONS

Bachelor of Arts : Year 2
Bachelor of Science : Year 1
Bachelor of Science : Year 2
Bachelor of Science : Year 3
Master of Mathematics : Year 2
Master of Physics : Year 2
Master of Physics : Year 4

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two hours and a half

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B. The marks shown against the questions, or parts of questions, indicate their relative weights. The total of marks available in Section A is 55.

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y^2}{x^2 + 4x + 3},$$

putting your answer in the form $y = f(x)$.

[5 marks]

2. Solve the initial value problem

$$xy\frac{dy}{dx} = 3x^2 + y^2 \; ; \quad y(1) = 2 \; .$$

[5 marks]

3. Solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 2x - y \\ \frac{dy}{dt} &= -4x + 2y \end{aligned}$$

given the initial conditions $x(0) = 2, y(0) = 1$.

[9 marks]

4. The Laplace transform of the function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt \, .$$

Show that

(i)

$$\mathcal{L}\left\{t f(t)\right\} = -\frac{d}{ds} F(s) \, ,$$

[3 marks]

(ii)

$$\mathcal{L}\{e^{kt}\} = \frac{1}{s-k} \, .$$

[2 marks]

Hence or otherwise find

$$\mathcal{L}^{-1}\{(s+2)^{-3}\}.$$

[5 marks]

5. The function $u(x, y)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} + 4xy \frac{\partial u}{\partial y} = 0 \, .$$

(i) Find the characteristic curves of this equation.

[5 marks]

(ii) Hence find the solution to the boundary value problem

$$u = \sin y \quad \text{when} \quad x = 0 \, .$$

[4 marks]

6. (i) State the definition of the Heaviside function $H(t)$, and find an expression for the Laplace transform

$$\mathcal{L}\{H(t-a)f(t-a)\}$$

[3 marks]

in terms of $\mathcal{L}\{f(t)\}$.

(ii) By using the Laplace transform or otherwise, solve the differential equation

$$\frac{dy}{dt} + 3y = 2H(t-2)$$

with the initial condition $y(0) = 1$. Sketch the solution.

[5 marks]

7. (i) Write down the Cauchy-Riemann equations connecting a function $u(x, y)$ to its conjugate harmonic function $v(x, y)$.

Show that the function

$$u(x, y) = 4\tan^{-1}\left(\frac{y}{x}\right)$$

satisfies the two-dimensional Laplace's equation.

Hint: Remember $\frac{d}{dt}\tan^{-1} t = 1/(t^2 + 1)$.

[4 marks]

(ii) Find $v(x, y)$, the conjugate harmonic function corresponding to $u(x, y)$ in part (i).

[5 marks]

SECTION B

8. The equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = e^{-t}$$

has the initial conditions: $y(0) = 2, y'(0) = 0$.

(i) Find the solution of this problem without using the Laplace transform.

[7 marks]

.

(ii) Laplace transform the equation and find the value of $Y(s)$, the Laplace transform of $y(t)$. Hence find the solution $y(t)$, stating explicitly each inverse Laplace transform you use.

[8 marks]

9. Find the general solution of the system of differential equations

$$\frac{dx}{dt} = 2x - 2y + 1 + e^t ,$$

$$\frac{dy}{dt} = -8x + 2y + e^t .$$

[15 marks]

10. (i) Sketch the graph of the function $g(t)$ where:

$$g(t) = 2 - 2\,|\cos t|$$

What is its period?

[5 marks]

(ii) Calculate the Fourier series for $g(t)$.

Hint: Remember that $\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$ for any A and B.

[10 marks]

11. A function $u(x, y)$ satisfies Laplace's equation in the rectangle $-a < x < a$, $0 < y < b$ together with the homogeneous boundary conditions

$$u(x, 0) = u(x, b) = 0, \quad -a < x < a$$

on the boundaries $y = 0$ and $y = b$.

(i) Show that the separable solutions of this boundary value problem are

$$u_n = \sin\left(\frac{n\pi y}{b}\right)\left[C_n \cosh\left(\frac{n\pi x}{b}\right) + D_n \sinh\left(\frac{n\pi x}{b}\right)\right]$$

where n is an integer and C_n and D_n are constants.

[8 marks]

(ii) Find the solution to this problem, i.e. find C_n and D_n, given that $u(x, y)$ satisfies the boundary conditions

$$u(-a, y) = -1, \;\; u(a, y) = 1, \;\; 0 < y < b$$

on $x = -a$ and $x = a$.

[7 marks]

12. The temperature $u(\theta, t)$ in a metal ring obeys the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial \theta^2} ,$$

where the angular coordinate θ will be chosen to run from $-\pi$ to π.

(i) By considering the separable solutions of the heat equation show that the general solution to the heat equation in the ring can be written as

$$u(\theta, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\kappa n^2 t} \left[a_n \cos n\theta + b_n \sin n\theta\right]$$

where n is an integer and a_n and b_n are constants.

[8 marks]

(i) Initially the temperature is 100 °C for $-\pi/4 < \theta < \pi/4$ and 0 °C in the rest of the ring. Find the temperature distribution at later times.

[7 marks]