SECTION A

1. *By forming an exact differential, or otherwise, find the general solution of the ordinary differential equation*

$$\frac{dy}{dx} = \frac{-(y^2 - 2x)}{2xy} \ ,$$

leaving your answer in the form $y = f(x)$.

[5 marks]

Solution [Homework]

Rearranging gives

$$(y^2 - 2x)\ dx + 2xy\ dy = 0.$$

This is an exact differential, as there is a function $u(x, y)$ with

$$\frac{\partial u}{\partial x} = y^2 - 2x, \quad \frac{\partial u}{\partial y} = 2xy; \qquad \frac{\partial^2 u}{\partial x \partial y} = 2y.$$

Then

$$u(x, y) = \int y^2 - 2x\ dx = xy^2 - x^2 + k(y)$$

for some function $k(y)$, which turns out to be 0. So $xy^2 - x^2 = c$, giving solution

$$y(x) = \sqrt{\frac{c + x^2}{x}} \ .$$

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2. *Solve the initial value problem*

$$\frac{dy}{dx} + 2xy = x \; ; \qquad y(0) = 1 \; .$$

[4 marks]

Solution [Homework]
This is a linear equation. Multiply by the integrating factor

$$\mu x = e^{\int 2x\, dx} = e^{x^2}$$

to find

$$\frac{d}{dx}\left(e^{x^2} y\right) = e^{x^2}\frac{dy}{dx} + 2xe^{x^2} y = xe^{x^2}$$

Integrating the right hand side gives

$$e^{x^2} y(x) = e^{x^2}/2 + c$$

so

$$y(x) = ce^{-x^2} + 1/2.$$

The initial conditions give

$$1 = c + 1/2$$

so $c = 1/2$ and hence the solution is

$$y(x) = \frac{e^{x^2} + 1}{2} \; .$$

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3. *Find the general solution of the ordinary differential equation*

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - 3y = 5x^2 \, .$$

[9 marks]

Solution [Homework]
The complemenary (homogeneous) equation is

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - 3y = 0.$$

Try solution $y = x^m$. Subsituting into the equation gives

$$m(m-1)x^m + 3mx^m - 3x^m = 0$$

so

$$m^2 + 2m - 3 = 0$$

giving roots $m = 1$ and $m = -3$. Hence the complementary solution is

$$y_C(x) = c_1 x + c_2/x^3.$$

For a particular solution, try $y_t(x) = \alpha x^2$. Substituting into the equation gives

$$2\alpha x^2 + 6\alpha x^2 - 3\alpha x^2 = 5x^2$$

which gives $5\alpha x^2 = 5x^2$, so $\alpha = 1$. The general solution is therefore

$$y(x) = c_1 x + c_2/x^3 + x^2$$

This question may also be solved by using the substitution $x = e^t$.

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4. *The Laplace transform of a function $f(t)$ is defined by*

$$\mathcal{L}\left\{f(t)\right\} = \widetilde{f}(s) = \int_0^\infty e^{-st} f(t)\, dt\ .$$

(i) *Show that*

$$\mathcal{L}\left\{f'(t)\right\} = s\widetilde{f}(s) - f(0)\ .$$

[2 marks]

Solution [Bookwork]

$$\begin{aligned}\mathcal{L}\left\{f'(t)\right\} &= \int_0^\infty f'(t)e^{-st}\, dt \;=\; \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty f(t)\,\left(-se^{-st}\right)\, dt \\ &= -f(0) + s\int_0^\infty f(t)e^{-st}\, dt \;=\; s\widetilde{f}(s) - f(0)\ .\end{aligned}$$

(ii) *Find a formula for $\mathcal{L}\left\{f''(t)\right\}$.*

[3 marks]

Solution [Bookwork]

$$\mathcal{L}\left\{f''(t)\right\} = s\mathcal{L}\left\{f'(t)\right\} - f'(0) = s^2\widetilde{f}(s) - sf(0) - f'(0).$$

(The question can also be done by computing $\int_0^\infty f''(t)\, e^{-st}\, dt$ directly.)

(iii) *Compute the Laplace transforms of* $\sin(at)$ *and* $\cos(at)$.

[4 marks]

Solution [Bookwork]

The second derivative of $\sin(at)$ is $-a^2\sin(at)$, so

$$-a^2\mathcal{L}\left\{\sin(at)\right\} = s^2\mathcal{L}\left\{\sin(at)\right\} - a$$

giving

$$\mathcal{L}\left\{\sin(at)\right\} = \frac{a}{s^2 + a^2}\ .$$

The derivative of $\sin(at)$ is $a\cos(at)$, so

$$a\mathcal{L}\left\{\cos(at)\right\} = s\mathcal{L}\left\{\sin(at)\right\}$$

giving

$$\mathcal{L}\left\{\cos(at)\right\} = \frac{s}{a}\,\frac{a}{s^2 + a^2} = \frac{s}{s^2 + a^2}\ .$$

(These can also be computed by calculating the Laplace transforms of $e^{\pm iat}$.)

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5. *Calculate the Fourier cosine series of period 2π for the function $f(x)$ defined for $0 < x < \pi$ by*

$$f(x) = x^2 .$$

[7 marks]

Solution [Homework]

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi f(x)\,dx \\ &= \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3} \\ a_0 &= \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi x^2\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n}\right]_0^\pi \\ &= \frac{2}{\pi}\frac{2\pi\cos(n\pi)}{n^2} = \frac{4(-1)^n}{n^2} \end{aligned}$$

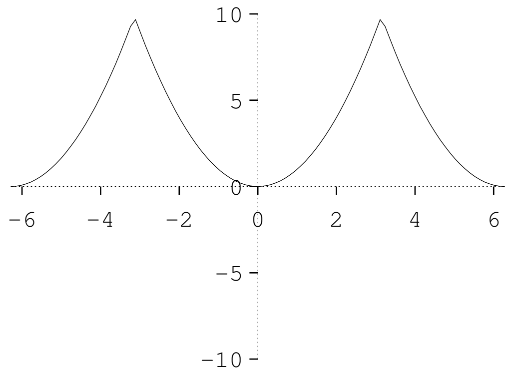
So the Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n^2} .$$

Sketch the graph of this cosine series for $-2\pi < x < 2\pi$.

[2 marks]

Solution [Bookwork]



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6. *The function $u(x,y)$ satisfies the first order partial differential equation*

$$2x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = u$$

in the domain $x > 0$, $y > 0$.

Show the the family of characteristics of the partial differential equation may be represented by

$$x = x_0 e^{2t}\ , \qquad y = y_0 e^{t}\ .$$

[3 marks]

Solution [Homework]

The characteristics are given by the differential equations

$$\frac{dx}{dt} = 2x, \qquad \frac{dy}{dt} = y$$

which have solutions

$$x = x_0 e^{2t} \quad \text{and } y = y_0 e^{t}$$

respectively.

Find the solution of the equation with $u = x$ when $y = x^2$.

[6 marks]

Solution [Homework]

Parameterising the boundary curve by s gives $x_0 = s$, $y_0 = s^2$ and $u_0 = s$. u is constant on the characteristic curves, so

$$x = se^{2t}, \quad y = s^2 e^t, \qquad u = s$$

Then $e^t = y/s^2$ so $e^{2t} = y^2/s^4$, hence $x = y^2/s^3$ and $s^3 = y^2/x$. Therefore the solution is

$$u(x,y) = \sqrt[3]{y^2/x}$$

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7. *The function* $u(x,t) = F(x)e^{-\lambda^2\kappa t}$, *where* κ *and* λ *are positive constants, is a nontrivial solution of the heat equation*

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \ .$$

Show that $F(x)$ *satisfies the ordinary differential equation*

$$F'' + \lambda^2 F = 0 \ .$$

[4 marks]

Solution [Bookwork]
Substituting into the heat equation gives

$$-\lambda^2\kappa F(x)e^{-\lambda^2\kappa t} = \kappa F''(x)e^{-\lambda^2\kappa t} \ .$$

Cancel $\kappa e^{-\lambda^2\kappa t}$ from each side to give

$$-\lambda^2 F(x) = F''(x)$$

which implies

$$F''(x) + \lambda^2 F(x) = 0.$$

Given that u *also satisfies the boundary conditions*

$$u(0,t) = u(L,t) = 0 \ ,$$

show that the possible values of λ *are* $n\pi/L$, *where* n *is a positive integer, and find the corresponding functions* $F(x)$.

[4 marks]

Solution [Bookwork]
The general solution for $F(x)$ is

$$F(x) = A\sin(\lambda x) + B\cos(\lambda x).$$

The boundary conditions give $F(0) = F(L) = 0$. Now

$$F(0) = A\sin(0) + B\cos(0) = B = 0$$

and then

$$F(L) = A\sin(\lambda L) = 0$$

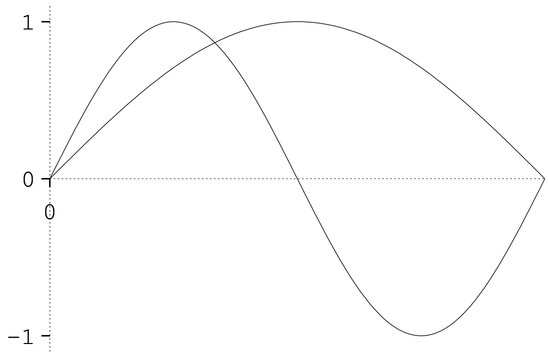
so either $A = 0$ or $\sin(\lambda L) = 0$. $A = 0$ gives the trivial solution, so $\sin(\lambda L) = 0$ which means $\lambda L = n\pi$ or $\lambda = n\pi/L$ for some positive integer n. Thus

$$\lambda = n\pi/L \quad \text{and } F_n(x) = \sin\left(\frac{n\pi x}{L}\right) \ .$$

Sketch $F(x)$ on the interval $0 \leq x \leq L$ for $n = 1$ and $n = 2$.

[2 marks]

Solution [Bookwork]



SECTION B

8. *Find the particular solution of the following system of equations, using the initial conditions* $x(0) = 1$, $y(0) = 0$.

$$\frac{dx}{dt} = 4x - y + 11t\ ,$$
$$\frac{dy}{dt} = 5x + 2y + 30t\ .$$

[15 marks]

Solution [Homework]
Write in vector form $d\mathbf{y}/dt = \mathrm{A}\mathbf{y} + \mathbf{f}(t)$ where

$$\mathrm{A} = \begin{pmatrix} 4 & -1 \\ 5 & 2 \end{pmatrix}, \quad \mathbf{f}(t) = t\begin{pmatrix} 11 \\ 30 \end{pmatrix} = t\mathbf{b}, \qquad \mathbf{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

First find the solution to the complementary homogeneous equation $d\mathbf{y}/dx = \mathrm{A}\mathbf{y}$. Try $\mathbf{y}(x) = e^{\lambda x}\mathbf{v}$. Then $\mathrm{A}\mathbf{v} = \lambda\mathbf{v}$, so λ is an eigenvalue of A and satisfies

$$\det(\mathrm{A} - \lambda \mathrm{I}) = (4 - \lambda)(2 - \lambda) - (-1)5 = \lambda^2 - 6\lambda + 13 = (\lambda - 3)^2 + 4 = 0.$$

The eigenvalues are complex, $\lambda = 3 \pm 2$, and the eigenvector corresponding to $\lambda = 3+2i$ is given by

$$\begin{pmatrix} 1-2i & -1 \\ 5 & -1-2i \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$$

or a scalar multiple. The real complementary solution is given by taking linear combinations of the real and imaginary parts of $e^{\lambda t}\mathbf{v}$, and is

$$\mathbf{y}(t) = c_1 e^{3t}(\cos(2t)\mathbf{u} - \sin(2t)\mathbf{w}) + c_2 e^{3t}(\sin(2t)\mathbf{u} + \cos(2t)\mathbf{w})$$

where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

are the real and imaginary parts of $\mathbf{v}$ respectively.

[6 marks for this part]

Try particular solution $\mathbf{y}(t) = t\mathbf{u}_1 + \mathbf{u}_0$ Substituting into the equation gives

$$\mathbf{u}_1 = t\mathrm{A}\mathbf{u}_1 + \mathrm{A}\mathbf{u}_0 + t\mathbf{b},$$

so

$$\mathrm{A}\mathbf{u}_1 = -\mathbf{b} \quad \text{and } \mathrm{A}\mathbf{u}_0 = \mathbf{u}_1.$$

To solve these equations, invert A. Then

$$\mathbf{u}_1 = -\mathrm{A}^{-1}\mathbf{b} = \frac{-1}{13}\begin{pmatrix} 2 & 1 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} 11 \\ 30 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

and

$$\mathbf{u}_0 = \mathrm{A}^{-1}\mathbf{u}_1 = \frac{1}{13}\begin{pmatrix} 2 & 1 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

So the general solution is

$$\begin{aligned} x(t) &= c_1 e^{3t}\cos(2t) + c_2 e^{3t}\sin(2t) - 4t - 1, \\ y(t) &= c_1 e^{3t}(\cos(2t) + 2\sin(2t)) + c_2 e 3t(\sin(2t) - 2\cos(2t)) - 5t. \end{aligned}$$

[6 marks for this part]

To satisfy the initial conditions, we require

$$\begin{aligned} c_1 - 1 &= 1, \\ c_1 - 2c_2 &= 0. \end{aligned}$$

This gives $c_1 = 2$ and $c_2 = 1$, so

$$\begin{aligned} x(t) &= 2e^{3t}\cos(2t) + e^{3t}\sin(2t) - 4t - 1, \\ y(t) &= 5e^{3t}\sin(2t) - 5t. \end{aligned}$$

[3 marks for this part]

(This problem can also be solved by substituting for one variable into the other equation or using the Laplace transform.)

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9. *The function $y(t)$ satisfies the ordinary differential equation*

$$\frac{d^2y}{dt^2} + \delta\,\frac{dy}{dt} + 4y = g(t)$$

where δ is a constant.

(a) *Suppose $\delta = 5$, and the initial conditions for $y(t)$ are $y(0) = y'(0) = 0$. Find a function $h(t)$ such that the solution to the initial value problem is*

$$y(t) = \int_0^t h(\tau) g(t-\tau)\, d\tau.$$

[8 marks]

Solution [Class]

Taking the Laplace transform gives

$$s^2Y(s) + 5sY(s) + 4Y(s) = G(s)$$

so

$$Y(s) = \frac{G(s)}{s^2+5s+4} = \frac{G(s)}{(s+1)(s+4)}.$$

[3 marks for this part]

Express $1/((s+1)(s+4))$ as a partial fraction

$$\frac{A}{s+1} + \frac{B}{s+4}\ .$$

The coefficients A and B satisfy

$$A(s+4) + B(s+1) = 1$$

so, setting $s=-1$ we find $B = 1/3$ and setting $s = -4$ we find $A = -1/3$. The inverse transform of $1/(s+a)$ is e^{-at}, so

$$\mathcal{L}^{-1}\left\{\frac{1}{3(s+4)} - \frac{1}{3(s+1)}\right\} = \frac{e^{-4t}}{3} - \frac{e^{-t}}{3}\ .$$

[3 marks for this part]

Then

$$\begin{aligned} y(t) &= \int_{\tau=0}^{t} \left(\frac{e^{-4(t-\tau)}}{3} - \frac{e^{-(t-\tau)}}{3}\right) g(\tau)\, d\tau \\ &= \frac{e^{-4t}}{3}\int_{\tau=0}^{t} e^{4\tau} g(\tau)\, d\tau - \frac{e^{-t}}{3}\int_{\tau=0}^{t} e^{\tau} g(\tau)\, d\tau\ . \end{aligned}$$

[2 marks for this part]

(b) *Suppose now $0 < \delta \ll 1$ is very small. Write down the form of the particular integral when $g(t) = \sin(nt)$, where n is a positive integer. Hence, or otherwise, approximate the amplitude of the steady-state response of the system to the input $g(t) = \sin(nt)$.*

[7 marks]

Solution [Homework]

The particular integral has the form $y(t) = A\sin(nt) + B\cos(nt)$. Substuting into the equation gives

$$\big(-An^2 - \delta Bn + 4A\big)\sin(nt) + \big(-Bn^2 + \delta An + 4B\big)\cos(nt) = \sin(nt)$$

Equating coefficients of $\sin(nt)$ and $\cos(nt)$ gives

$$\begin{aligned}(4-n^2)A - \delta nB &= 1\\ (4-n^2)B + \delta nA &= 0\end{aligned}$$

[3 marks for this part]

As $\delta \approx 0$, if $n \neq 2$, then

$$A \approx \frac{1}{4-n^2} \quad \text{and} \quad B \approx 0$$

so the amplitude is approximately $1/(4-n^2)$.

If $n = 2$, then

$$A = 0 \quad \text{and} \quad B = \frac{1}{2\delta}$$

so the amplitude is approximately $1/2\delta$, which is very large.

[4 marks for this part]

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10. *Suppose the Laplace transform of $f(t)$ is $F(s)$. Show that the Laplace transform of $H(t-a)f(t-a)$ is $e^{-as}F(s)$ if $a \geq 0$.*

[4 marks]

Solution [Bookwork]

$$\begin{aligned}\int_0^\infty H(t-a)f(t-a)e^{-st}\,dt &= \int_a^\infty f(t-a)e^{-st}\,dt = \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau \\ &= e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = e^{-as}F(s).\end{aligned}$$

The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + 2xu = 0$$

in the domain $x > 0$, $t > 0$, with boundary conditions

$$u(x,0) = 0, \quad u(0,t) = \sin(3t).$$

By taking the Laplace transform of $u(x,t)$ with respect to t, or otherwise, determine the function $u(x,t)$.

[11 marks]

Solution [Homework]

Taking the Laplace transform gives

$$\frac{d\widetilde{u}(x,s)}{ddx} + (s+2x)\widetilde{u}(x,s) = 0.$$

[3 marks for this part]

Solving this equation

$$\begin{aligned}\int \frac{d\widetilde{u}}{\widetilde{u}} &= \int -2x - s\,dx \\ \ln|\widetilde{u}(x,s)| &= -x^2 - sx + c(s) \\ \widetilde{u}(x,s) &= A(s)e^{-x^2}e^{-sx}\end{aligned}$$

[3 marks for this part]

$A(s) = \widetilde{u}(0,s)$, so $\widetilde{u}(x,s) = e^{-x^2}\widetilde{u}(0,s)e^{-sx}$. Now $\widetilde{u}(0,s)e^{-sx}$ is the Laplace transform of $u(0,t-x)H(t-x)$, so

$$\begin{aligned}u(x,t) &= e^{-x^2}u(0,t-x)H(t-x) \\ &= e^{-x^2}\sin(3(t-x))H(t-x).\end{aligned}$$

[5 marks for this part]

(This equation may also be solved by using the method of characteristics.)

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11. *Find the solution of the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(a) *in an infinite string, with initial conditions*

$$u(x,0) = \sin(x) \quad \textit{and} \quad u_t(x,0) = 0.$$

[4 marks]

Solution [Homework]
Since the string is infinite, use D'Alembert's solution of the wave equation,

$$u(x,t) = F(x+ct) + G(x-ct).$$

The initial conditions give

$$F(x) + G(x) = \sin(x) \quad \text{and} \quad cF(x) - cG(x) = 0,$$

so $F(x) = G(x) = \sin(x)/2$. Thus the solution is

$$u(x,t) = \frac{\sin(x+ct) + \sin(x-ct)}{2}$$

(b) *in a string of length L with boundary conditions $u(0,t) = u(L,t) = 0$, and initial conditions*

$$u(x,0) = f(x) = \begin{cases} x & \text{if } 0 \le x < L/2 \\ L - x & \text{if } L/2 < x \le L \end{cases} \quad \textit{and} \quad u_t(x,0) = 0.$$

[11 marks]

Solution [Homework]
The general form of the solution with the given boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right).$$

The boundary conditions give

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

so to solve the problem we need to find the Fourier sine series of the function $f(x)$.

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx \\ &= \frac{2}{L}\left(\int_0^{L/2} x\sin(n\pi x/L)\,dx + \int_{L/2}^{L} (L-x)\sin(n\pi x/L)\,dx\right) \end{aligned}$$

Setting $y = L - x$ in the second integral, we find

$$\begin{aligned}
\int_{L/2}^{L} (L-x)\sin(n\pi x/L)\,dx &= \int_{L/2}^{0} y\sin(n\pi - n\pi y/L)\,(-1)dy \\
&= \int_{0}^{L/2} y\cos(n\pi)\sin(-n\pi y/L)\,dy \\
&= -1^{n+1}\int_{0}^{L/2} y\sin(n\pi y/L)\,dy
\end{aligned}$$

so $B_n = 0$ if n is even. If n is odd,

$$\begin{aligned}
B_n &= \frac{4}{L}\int_{0}^{L/2} x\sin(n\pi x/L) \\
&= \frac{4}{L}\left[\frac{-Lx\cos(n\pi x/L)}{n\pi} + \frac{L^2\sin(n\pi xL)}{n^2\pi^2}\right]_0^{L/2} \\
&= \frac{4}{L}\left(\frac{L^2(1-\cos(n\pi/2))}{2\pi n} + \frac{L^2\sin(n\pi/2)}{n^2\pi^2}\right).
\end{aligned}$$

Setting $n = 2m - 1$, we have

$$B_{2m+1} = \frac{2L}{n\pi} + \frac{(-1)^m 4L}{n^2\pi^2}$$

So

$$u(x,t) = 2L\sum_{m=1}^{\infty}\left(\frac{(2m+1)\pi + 2(-1)^m}{(2m+1)^2\pi^2}\right)\sin\left(\frac{(2m+1)\pi x}{L}\right)\cos\left(\frac{(2m+1)\pi ct}{L}\right)\ .$$

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12. *Show that the function*

$$u(x,y) = x^3 - 3xy^2 + x$$

satisfies Laplace's equation.

[3 marks]

Solution [Example]

Compute

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1, \quad \frac{\partial^2 u}{\partial x^2} = 6x, \qquad \frac{\partial u}{\partial y} = -6xy, \quad \frac{\partial^2 u}{\partial y^2} = -6x,$$

so

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

In polar coordinates, Laplace's equation can be expressed as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \ .$$

Show that the functions

$$u_n(r,\theta) = A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)$$

are solutions of Laplace's equation.

[4 marks]

Solution [Class]

Compute the derivatives

$$\begin{aligned} \frac{\partial u_n}{\partial r} &= nA_n r^{n-1}\cos(n\theta) + nB_n n r^{n-1}\sin(n\theta) \\ \frac{\partial^2 u_n}{\partial r^2} &= n(n-1)A_n r^{n-2}\cos(n\theta) + n(n-1)B_n r^{n-2}\sin(n\theta) \\ \frac{\partial^2 u_n}{\partial \theta^2} &= -n^2 A_n r^n \cos n\theta - n^2 B_n r^n \sin(n\theta) \end{aligned}$$

Substituting into Laplace's equation gives

$$\begin{aligned} \frac{\partial^2 u_n}{\partial r^2} + \frac{1}{r}\frac{\partial u_n}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_n}{\partial \theta^2} &= A_n\left(n(n-1) + n - n^2\right) r^{n-2}\cos(n\theta) \\ &\qquad + B_n\left(n(n-1) + n - n^2\right) r^{n-2}\sin(n\theta) \\ &= 0 \end{aligned}$$

Why are the functions

$$r^{5/2}\cos(5\theta/2) \qquad \textit{and} \qquad r^{-3}\sin(3\theta)$$

not *solutions of Laplace's equation in the disc* $r < a$*?*

[2 marks]

Solution [Bookwork]

The function $r^{5/2}\cos(5\theta/2)$ is not a solution since if is not 2π-periodic in θ; if $\theta = 0$, we have $r^{1/2}$ and if $\theta = 2\pi$ we have $-r^{1/2}$. The function $r^{-3}\cos(3\theta)$ is not a solution since it is unbounded as $r \to 0$.

Find the solution of Laplace's equation in the disc $r < 2$ *which satisfies the boundary conditions*

$$u(r,\theta) = f(\theta) = \begin{cases} 1 & \text{if } 0 < \theta < \pi \\ -1 & \text{if } -\pi < \theta < 0. \end{cases}$$

on the circle $r = 2$*.*

[6 marks]

Solution [Homework]

The general solution is

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

The boundary conditions $u(2,\theta) = f(\theta)$ can be used to compute the coefficients A_n and B_n. $f(\theta)$ is an odd function, so we only need the Fourier sine series of f, which has coefficients

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \sin(n\theta)\, d\theta = \frac{2}{\pi}\left[\frac{-\cos(n\theta)}{n}\right]_0^{\pi} = \frac{2(1-\cos(n\pi))}{n\pi} \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \dfrac{4}{(2m-1)\pi} & \text{if } n = 2m-1. \end{cases} \end{aligned}$$

Now, $B_n = b_n/2^n$, so

$$u(r,\theta) = \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{(r/2)^{2m-1}\sin((2m-1)\theta)}{2m-1}.$$
