PAPER CODE NO.
MATH224

THE UNIVERSITY
of LIVERPOOL

SUMMER 2002 EXAMINATIONS

Bachelor of Arts : Year 2
Bachelor of Science : Year 1
Bachelor of Science : Year 2
Bachelor of Science : Year 3
Master of Mathematics : Year 2
Master of Physics : Year 2

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two hours and a half

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B. The marks shown against the questions, or parts of questions, indicate their relative weights. The total of marks available in Section A is 55.

SECTION A

1. By forming an exact differential, or otherwise, find the general solution of the ordinary differential equation

$$\frac{dy}{dx} = \frac{-(y^2 - 2x)}{2xy} ,$$

leaving your answer in the form $y = f(x)$.

[5 marks]

2. Solve the initial value problem

$$\frac{dy}{dx} + 2xy = x ; \qquad y(0) = 1 .$$

[4 marks]

3. Find the general solution of the ordinary differential equation

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - 3y = 5x^2 .$$

[9 marks]

4. The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}\left\{f(t)\right\} = \widetilde{f}(s) = \int_0^\infty e^{-st} f(t)\, dt .$$

(i) Show that

$$\mathcal{L}\left\{f'(t)\right\} = s\widetilde{f}(s) - f(0) .$$

[2 marks]

(ii) Find a formula for $\mathcal{L}\left\{f''(t)\right\}$.

[3 marks]

(iii) Compute the Laplace transforms of $\sin(at)$ and $\cos(at)$.

[4 marks]

5. Calculate the Fourier cosine series of period 2π for the function $f(x)$ defined for $0 < x < \pi$ by

$$f(x) = x^2 \, .$$

[7 marks]

Sketch the graph of this cosine series for $-2\pi < x < 2\pi$.

[2 marks]

6. The function $u(x, y)$ satisfies the first order partial differential equation

$$2x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$$

in the domain $x > 0$, $y > 0$.
Show the the family of characteristics of the partial differential equation may be represented by

$$x = x_0 e^{2t} \, , \qquad y = y_0 e^t \, .$$

[3 marks]

Find the solution of the equation with $u = x$ when $y = x^2$.

[6 marks]

7. The function $u(x, t) = F(x) e^{-\lambda^2 \kappa t}$, where κ and λ are positive constants, is a nontrivial solution of the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \, .$$

Show that $F(x)$ satisfies the ordinary differential equation

$$F'' + \lambda^2 F = 0 \, .$$

[4 marks]

Given that u also satisfies the boundary conditions

$$u(0, t) = u(L, t) = 0 \, ,$$

show that the possible values of λ are $n\pi/L$, where n is a positive integer, and find the corresponding functions $F(x)$.

[4 marks]

Sketch $F(x)$ on the interval $0 \leq x \leq L$ for $n = 1$ and $n = 2$.

[2 marks]

SECTION B

8. Find the particular solution of the following system of equations, using the initial conditions $x(0) = 1$, $y(0) = 0$.

$$\begin{aligned}\frac{dx}{dt} &= 4x - y + 11t\ , \\ \frac{dy}{dt} &= 5x + 2y + 30t\ .\end{aligned}$$

[15 marks]

9. The function $y(t)$ satisfies the ordinary differential equation

$$\frac{d^2y}{dt^2} + \delta\,\frac{dy}{dt} + 4y = g(t)$$

where δ is a constant.

(a) Suppose $\delta = 5$, and the initial conditions for $y(t)$ are $y(0) = y'(0) = 0$. Find a function $h(t)$ such that the solution to the initial value problem is

$$y(t) = \int_0^t h(\tau)g(t-\tau)\ d\tau.$$

[8 marks]

(b) Suppose now $0 < \delta \ll 1$ is very small. Write down the form of the particular integral when $g(t) = \sin(nt)$, where n is a positive integer. Hence, or otherwise, approximate the amplitude of the steady-state response of the system to the input $g(t) = \sin(nt)$.

[7 marks]

10. Suppose the Laplace transform of $f(t)$ is $F(s)$. Show that the Laplace transform of $H(t-a)f(t-a)$ is $e^{-as}F(s)$ if $a \geq 0$.

[4 marks]

The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + 2xu = 0$$

in the domain $x > 0$, $t > 0$, with boundary conditions

$$u(x,0) = 0, \quad u(0,t) = \sin(3t).$$

By taking the Laplace transform of $u(x,t)$ with respect to t, or otherwise, determine the function $u(x,t)$.

[11 marks]

11. Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(a) in an infinite string, with initial conditions

$$u(x,0) = \sin(x) \quad \text{and} \quad u_t(x,0) = 0.$$

[4 marks]

(b) in a string of length L with boundary conditions $u(0,t) = u(L,t) = 0$, and initial conditions

$$u(x,0) = f(x) = \begin{cases} x & \text{if } 0 \leq x < L/2 \\ L - x & \text{if } L/2 < x \leq L \end{cases} \quad \text{and} \quad u_t(x,0) = 0.$$

[11 marks]

12. Show that the function

$$u(x,y) = x^3 - 3xy^2 + x$$

satisfies Laplace's equation.

[3 marks]

In polar coordinates, Laplace's equation can be expressed as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\ .$$

Show that the functions

$$u_n(r,\theta) = A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)$$

are solutions of Laplace's equation.

[4 marks]

Why are the functions

$$r^{5/2}\cos(5\theta/2) \quad \text{and} \quad r^{-3}\sin(3\theta)$$

not solutions of Laplace's equation in the disc $r < a$?

[2 marks]

Find the solution of Laplace's equation in the disc $r < 2$ which satisfies the boundary conditions

$$u(r,\theta) = f(\theta) = \begin{cases} 1 & \text{if } 0 < \theta < \pi \\ -1 & \text{if } -\pi < \theta < 0. \end{cases}$$

on the circle $r = 2$.

[6 marks]