MATH224 Summer Exam

Solutions

1. Separate the variables:

$$\begin{aligned} \int \frac{dy}{y+5} &= \int (1+3x^4)dx \\ \ln(y+5) &= x + \frac{3x^5}{5} + C \\ \Rightarrow y+5 &= D\exp(x+3x^5/5) \end{aligned}$$

where $D = \exp(C)$ and C is a constant of integration.

2. Put $y = x^m$: then

$$\begin{aligned} 2m(m-1)+m-1 &= 0 \\ \Rightarrow (2m+1)(m-1) &= 0 \\ \Rightarrow m = -\frac{1}{2} \quad m &= 1 \\ \Rightarrow y &= Ax + \frac{B}{\sqrt(x)} \end{aligned}$$

where A and B are constants.

3.

$$\begin{aligned} \dot{x} + y &= 0 \\ 4\dot{x} - \dot{y} + 4x &= 0 \end{aligned}$$

Substituting the first of these equations into the second, we get

$$\ddot{x} + 4\dot{x} + 4x = 0$$

The auxiliary equation is

$$\begin{aligned} \lambda^2 + 4\lambda + 4 &= 0 \\ \Rightarrow \quad (\lambda + 2)^2 &= 0 \end{aligned}$$

Hence the general solution is

$$\begin{aligned} x &= Ate^{-2t} + Be^{-2t} \\ \Rightarrow \dot{x} &= Ae^{-2t}(1 - 2t) - 2Be^{-2t} \\ \Rightarrow \dot{x} &= Ae^{-2t}(1 - 2t) - 2Be^{-2t} \\ \Rightarrow y &= Ae^{-2t}(2t - 1) + 2Be^{-2t} \end{aligned}$$

From the initial conditions, we have

$$\begin{aligned} B &= 1 \\ A &= 2 \end{aligned}$$

Hence

$$\begin{aligned} x &= 2te^{-2t} + e^{-2t} \\ y &= 2e^{-2t}(2t - 1) + 2e^{-2t} \end{aligned}$$

4. (i) From the definition of the L.T.

$$\begin{aligned} \mathcal{L}\{H(t-k)\} &= \int_0^\infty e^{-st}\ H(t-k) \cdot dt \\ &= \int_k^\infty e^{-st} dt \\ &= \left[\frac{e^{-st}}{-s}\right]_k^\infty \\ &= \frac{e^{-ks}}{s} \end{aligned}$$

(ii) The function f(t) can be written

$$f(t) = \frac{1}{a}\left(H(t-k) - H(t-(k+a))\right)$$

For which the L.T. is

$$F(s) = \frac{e^{-ks}}{as} - \frac{e^{-(k+a)s}}{as}$$

5.

$$f(x) = a_0 + \sum_n a_n \cos(nx)$$

Since this is an even function, we have $b_n = 0$ Also, $a_0 = 0$ since this is the area under the graph from $-\pi$ to π.

We have

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{\pi}^{\pi} f(x)\cos(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)dx \\
&= \frac{1}{\pi}\left[\int_{0}^{\pi/2} -\cos(nx)dx + \int_{\pi/2}^{\pi} \cos(nx)dx\right] \\
&= \frac{1}{\pi}\left[\frac{-1}{n}[\sin(nx))]_0^{\pi/2} + \frac{1}{n}[\sin(nx))]_{\pi/2}^{\pi}\right] \\
&= \frac{2}{n\pi}\left[-\sin\left(\frac{nx}{2}\right)\right] \\
&= \frac{-2}{n\pi}\sin\left(\frac{nx}{2}\right)
\end{aligned}$$

6. Differentiating the given form for u we find

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{d^2 F}{dx^2}\exp(-\lambda^2 kt) \\
\frac{1}{k}\frac{\partial u}{\partial t} &= -\lambda^2 F(x)
\end{aligned}$$

Substituting these into the equation given yields

$$\frac{d^2 F}{dx^2} = -\lambda^2 F$$

The solution to this is of the form

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

to satisfy the boundary conditions $A = 0$ and $\sin(\lambda d) = 0$.

Therefore $\lambda d = n\pi$ where n is an integer. Hence $\lambda = n\pi/d$ giving the general solution

$$u(x) = \sum_{n=1} \sin(n\pi x/d)\exp(-n^2\pi^2 kt/d^2)$$

7. Laplace's equation in 2d is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

We have

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} = e^x \cos(y)$$

and

$$\frac{\partial u}{\partial y} = -e^x \cos(y)$$

Thus Laplace's equation is satisfied.

The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos(y)$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2} = -e^x \sin(y)$$

From the first of these we get

$$v = e^x \sin(y) + g(x)$$

with $g(x)$ an unknown function. From the second equation we get

$$\frac{dg}{dx} = 0$$

whence $g = C$, where C is a constant of integration.

8. (i)

$$\begin{aligned} \mathcal{L}(e^{at}) &= \int_0^\infty e^{-st}e^{at}\cdot dt \\ &= \left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a} \qquad \text{for } s > a \end{aligned}$$

(ii) Here we need to perform an integration by parts of $f'(t)$, which yields

$$\begin{aligned} &\int_0^\infty f'(t)e^{-st}dt \\ &= \left[fe^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt \\ &= -f(0) + s\mathcal{L}f(t) \end{aligned}$$

(iii)

$$\begin{aligned} \mathcal{L}(f''(t)) &= -f'(0) + s\mathcal{L}(f'(t)) \\ &= -f'(0) - sf(0) + s^2\mathcal{L}(f) \\ &= -f(0) + s\mathcal{L}f(t) \end{aligned}$$

Substituting the results of (ii) and (iii) above into the differential equation given, writing $Y(s) \equiv \mathcal{L}\{y()\}$, and inserting the initial conditions, one obtains

$$s^2Y(s) - s + 2 + 4(sY(s) - 1) + 3Y(s) = (s^2 + 4s + 3)Y(s) = s + 2 + \frac{1}{s+2}$$

Using partial fractions

$$\begin{aligned} Y(s) &= \frac{1}{s+3} + \frac{1}{s+1} - \frac{1}{s+2} \\ \Rightarrow\ y(t) &= \exp(-3t) + \exp(-t) - \exp(-2t) \end{aligned}$$

9. Firstly, we note that the function is even so $b_n = 0$

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{-T}^{T} |\sin\left(\frac{2\pi t}{T}\right)|dt \\ &= \frac{4}{\pi} \end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_{-T/2}^{T/2} |\sin(\pi x/T)| \cos(n2\pi x/T)dx \\
&= \frac{4}{T}\int_{0}^{T/2} \sin(\pi x/T) \cos(n2\pi x/T)dx \\
&= \frac{2}{T}\int_{0}^{T/2} \{\sin((2n+1)\pi x/T) - \sin((2n-1)\pi x/T)\}\, dx \\
&= \frac{2}{\pi}\left[-\frac{\cos((2n+1)\pi/2-1)}{2n+1} + \frac{\cos((2n-1)\pi/2-1}{2n-1}\right] \\
&= \frac{2}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) \\
&= \frac{-4}{\pi(2n+1)(2n-1)}
\end{aligned}
$$

If $n = 2m - 1$, the above is zero, so put $n = 2m$. Then

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\left[\frac{2}{2m+1} - \frac{2}{2m-1}\right] \\
&= \frac{-4}{\pi(2m+1)(2m-1)}
\end{aligned}
$$

Substituting into the definition of the Fourier series:

$$
F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi t}{T})
$$

yields the required result

$$
f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} \cos\left(\frac{n\pi t}{T}\right) .
$$

Now let a trial particular integral be

$$
x = b_0 + \sum b_m \cos\left(\frac{n\pi t}{T}\right)
$$

Then $b_0 = 2/pi$. Differentiating x twice and substituting into the equation, we get

$$
\left[-\left(\frac{n\pi}{T}\right)^2 + 1\right] b_m = \frac{-4}{\pi(2m+1)(2m-1)}
$$

which specifies b_m.

10. The characteristics are given by

$$\begin{aligned}\frac{dx}{dt} &= 1+y \qquad (1)\\ \frac{dy}{dt} &= y \qquad (2)\\ \frac{du}{dt} &= u+y \qquad (3)\end{aligned}$$

The boundary conditions are $x=0, y=s, u=s(1-s)$. s is a parameter which gives a position on the boundary and t is a parameter which gives a parameter on the characteristic.

From (2) we have

$$y = s\exp(t)$$

From (1) and (2)

$$d(x-y)/dt = 1 \Rightarrow x-y = t-s$$

So

$$x = s\exp(t)+t+s$$

From (3)

$$\frac{du}{dt} = u + s\exp(t)$$

Using an integrating factor, we get

$$\frac{d(u\exp(-t))}{dt} = s$$

Hence

$$u\exp(-t) = st + C = st + s(1-s) \Rightarrow u = s\exp(t+1-s) \Rightarrow u = y(1+x-y)$$

11.

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta}\end{aligned}$$

Hence

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \\
\frac{\partial^2}{\partial y^2} &= \frac{\partial^2}{\partial \xi^2} - 4\frac{\partial^2}{\partial \xi \partial \eta} + 4\frac{\partial^2}{\partial \eta^2} \\
\frac{\partial^2}{\partial x \partial y} &= \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \xi \partial \eta} - 2\frac{\partial^2}{\partial \eta^2}
\end{aligned}
$$

Therefore

$$4u_{xx} + 4u_{xy} + u_{yy} = 9u_{\xi\xi} + 0 + 0$$

and the equation is therefore parabolic.

Now solving for x and y, we have

$$3x = 2\xi + \eta \qquad\qquad 3y = \xi - \eta$$

Now

$$
\begin{aligned}
9(x^2 - xy - 2y^2) &= 4\xi^2 + 4\xi\eta + \eta^2 - (2\xi^2 - \xi\eta - \eta^2) - 2(\xi^2 - 2\xi\eta + \eta^2) \\
&= 0 + 9\xi\eta + 0
\end{aligned}
$$

so the equation is

$$
\begin{aligned}
9u_{\xi\xi} &= 9\xi\eta \\
\Rightarrow u_\xi &= \frac{\xi^2\eta}{2} + f(\eta) \\
\Rightarrow u &= \frac{1}{6}\frac{\xi^3\eta}{2} + \xi f(\eta) + g(\eta) \\
&= \frac{1}{6}\Big((x+y)^2(x-2y)\Big) + (x+y)f(x-2y) + g(x-2y)
\end{aligned}
$$

12. (i)

$$
\begin{aligned}
\mathcal{L}(H(t-a)f(t-a)) &= \int_0^\infty e^{-st}\, H(t-a)f(t-a)dt \\
&= \int_a^\infty e^{-st}\, f(t-a)dt
\end{aligned}
$$

Now let $\tau = t - a$ then

$$\begin{aligned}
\mathcal{L}(H(t-a)f(t-a)) &= \int_0^\infty e^{-s(\tau+a)}\ f(\tau)d\tau \\
&= e^{-as}\int_0^\infty e^{-s\tau}\ f(\tau)d\tau \\
&= e^{-as}F(s)
\end{aligned}$$

(ii) Taking the L.T of the equation we get

$$\tilde{u}'' - 2s\tilde{u}' + s^2\tilde{u} = 0$$

(iii) The boundary conditions are

$$\tilde{u}(x,s) = \int_0^\infty e^{-st}u(x,t)dt$$

Thus

$$\tilde{u}(0,s) = 0$$

$$\begin{aligned}
\tilde{u}(1,s) &= \int_0^\infty te^{-st}dt \\
&= \frac{-1}{s}\left[te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt \\
&= 0 - \frac{1}{s^2}\left[e^{-st}\right]_0^\infty \\
&= \frac{1}{s^2}
\end{aligned}$$

(iv)

$$\tilde{u}'' - 2s\tilde{u}' + s^2\tilde{u} = 0$$

Auxiliary equation is

$$m^2 - 2sm + s^2 = 0$$

Double root, solution:

$$\tilde{u} = (Ax + B)e^{sx}$$

Boundary conditions $x = 0 \Rightarrow B = 0$; $x = 1 \Rightarrow A = e^s/s^2$. Hence

$$\tilde{u} - \frac{e^s x}{s^2}e^{sx} = \frac{xe^{s(x+1)}}{s^2}$$

Therefore

$$u(x,t) = x\,(t + x + 1)\,H(t - (x+1))$$