SECTION A

1. Find the general solution of

$$\frac{dy}{dx} = (1 + 3x^4)(y + 5)$$

leaving your answer in the form $y = f(x)$. [4 marks]

2. Solve the differential equation

$$2x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

[6 marks]

3. Solve the system of differential equations for $x = x(t)$ and $y = y(t)$

$$\begin{aligned} \dot{x} + y &= 0 \\ 4\dot{x} - \dot{y} + 4x &= 0 \end{aligned}$$

given the initial conditions $x(0) = 1, y(0) = 0$. [9 marks]

4. (i) Show that the Laplace transform of the Heaviside function $H(t - k)$ with $k > 0$ is given by

$$\mathcal{L}\{H(t-k)\} = \frac{e^{-ks}}{s}$$

[4 marks]

(ii) Calculate the Laplace transform of the function

$$f(t) = \begin{cases} 0 & 0 \le t < k \\ \frac{1}{a} & k \le t < k + a \\ 0 & t \ge k + a \end{cases}$$

[4 marks]

5. The function $f(x)$ is even and has period 2π; it also satisfies

$$f(x) = \begin{cases} -\frac{1}{2}\,, & 0 \le x < \pi/2 \\ \frac{1}{2}\,, & \pi/2 \le x \le \pi\,. \end{cases}$$

Sketch the graph of the function for $-2\pi \le x < 2\pi$ and find its Fourier series.
[10 marks]

6. Given $u(x,t) = F(x)\exp(-\lambda^2 kt)$, where k and λ are constants, satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

show that $F'' + \lambda^2 F = 0$. [4 marks]

Given that solutions of the above partial differential equation also satisfy the boundary conditions

$$u(0,t) = u(d,t) = 0$$

where d is a constant, find the possible values of λ. Hence find the general solution for this boundary value problem. [5 marks]

7. Show that the function $u(x,y) = e^x \cos(y)$ satisfies the two-dimensional Laplace's equation. [3 marks]

Write down the Cauchy-Riemann equations involving u and its conjugate harmonic function $v(x,y)$. Find $v(x,y)$. [6 marks]

SECTION B

8. Given that the Laplace transform of $f(t)$ is denoted $F(s)$, show that

(i)

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

[2 marks]

(ii)

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

[3 marks]

(iii)

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

[3 marks]

(iv) Given that $y(t)$ satisfies the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-2t}$$

and the boundary conditions $y = 1$ and $\frac{dy}{dt} = -2$ at $t = 0$, show that the Laplace transform of $y(t)$, denoted $Y(s)$ satisfies

$$(s+1)(s+3)Y(s) = \frac{1}{s+2} + s + 2\,.$$

Hence find $y(t)$. [7 marks]

9. The periodic function $f(t)$ is defined by

$$f(t) = |\sin\left(\frac{\pi t}{T}\right)|\,, \qquad -T < t \leq T$$

and $f(t+T) = f(t)$. Show that the Fourier series of $f(t)$ may be written

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n-1)}\cos\left(\frac{2n\pi t}{T}\right)\,.$$

[10 marks]

Find a particular integral of the ordinary differential equation

$$\frac{d^2x}{dt^2} + x = f(t)$$

assuming resonance does not occur. [5 marks]

10. The function $u(x,y)$ satisfies the first order partial differential equation

$$(1+y)\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u+y$$

in the domain $x>0,\ y>0$ and the boundary condition

$$u=y(1-y) \qquad \text{on} \qquad x=0\ .$$

Show that the family of characteristics of the partial differential equation may be represented by

$$x=t+se^t-s\ , \qquad y=se^t$$

where s and t are parameters whose significance you should explain. [7 marks]

Hence determine the function $u(x,y)$. [8 marks]

11. Use the transformation

$$\xi=x+y, \qquad \eta=x-2y$$

to reduce the partial differential equation

$$4u_{xx}+4u_{xy}+u_{yy}=9(x^2-xy-2y^2)$$

to canonical form. Hence or otherwise, classify the equation. [11 marks]

Find the general solution of this equation in terms of x and y. [4 marks]

12. (i) Writing $\tilde{f}(s)$ for the Laplace transform of $f(t)$ and $H(t-a)$ for the Heaviside (or unit step) function, show that the Laplace transform of $f(t-a)H(t-a)$ is

$$\tilde{f}(s)\exp(-as)\ .$$

[3 marks]

(ii) The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0\ , \qquad 0 < x < 1, \quad t > 0\ ,$$

and the initial and the boundary conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \frac{\partial u}{\partial x}(x,0) = 0\ ,$$

$$u(0,t) = 0, \quad u(1,t) = t\ .$$

Show that the Laplace transform of $u(x,t)$ with regard to t, denoted by $\tilde{u}$, satisfies the ordinary differential equation

$$\tilde{u}'' - 2s\tilde{u}' + s^2\tilde{u} = 0\ .$$

[5 marks]

(iii) Find the boundary conditions for $\tilde{u}$ at $x = 0$ and at $x = 1$. [2 marks]

(iv) Solve this equation for $\tilde{u}$ and hence find the function $u(x,t)$. [5 marks]