All problems are similar to homework and class examples, except where stated explicitly as bookwork.

1. **Question** *Find the general solutions of the differential equations:*

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}+(1+x)y^2=2x^3y^2,$$

Answer This is separable:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x^3-x-1}{x^2}y^2.$$

Separating the variables gives

$$\int\frac{\mathrm{d}y}{y^2}=\int\left(2x-\frac{1}{x}-\frac{1}{x^2}\right)\,\mathrm{d}x=-\frac{1}{y}=x^2-\ln|x|+\frac{1}{x}+C.$$

So the solution is

$$y=-\left(x^2-\ln|x|+1/x+C\right)^{-1}.$$

Question

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+(1+x^2)y=2x^3.$$

Answer This equation is linear

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1+x^2}{x}y=2x^2.$$

The integrating factor is

$$\exp\left(\int\left(x+\frac{1}{x}\right)\,\mathrm{d}x\right)=xe^{x^2/2},$$

(need only one solution so no arbitrary constant needed), multiplying through by it gives

$$xe^{x^2/2}\frac{\mathrm{d}y}{\mathrm{d}x}+(1+x^2)e^{x^2/2}y=\left(xe^{x^2/2}y\right)'=2x^3e^{x^2/2}.$$

Integrating this equation gives

$$xe^{x^2/2}y=\int 2x^3e^{x^2/2}\,\mathrm{d}x=4\int(x^2/2)e^{x^2/2}\,\mathrm{d}\left(x^2/2\right)=4(x^2/2-1)e^{x^2/2}+C.$$

hence

$$y=2x-\frac{4}{x}+\frac{C}{x}e^{-x^2/2}.$$

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2. **Question** *Solve the initial value problem:*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+11\frac{\mathrm{d}y}{\mathrm{d}x}+28y=28x^2+22x+30,\qquad y(0)=0,\qquad y'(0)=1.$$

Answer The solution $y=e^{mx}$ will satisfy the homogeneous equation if $m^2+11m+28=0$. This has roots $m=-4$ and $m=-7$ so the Complementary Function is $Ae^{-4x}+Be^{-7x}$. To find the particular integral we try $y=\alpha x^2+\beta x+\gamma$. We find

$$2\alpha+11(2\alpha x+\beta)+28(\alpha x^2+\beta x+\gamma)=28x^2+22x+30$$

This is satisfied if $\alpha=1$, $22\alpha+28\beta=22$, so that $\beta=0$ and $2\alpha+11\beta+28\gamma=30$, so that $\gamma=1$. The general solution is then

$$y=Ae^{-4x}+Be^{-7x}+x^2+1$$

From the initial conditions we get

$$y(0)=A+B+1=0,\qquad y'(0)=1=-4A-7B$$

Therefore $A=-2$ and $B=1$.

The solution is therefore $y=-2e^{-4x}+e^{-7x}+x^2+1$.

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$$(x^2+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-(x^2-2)\frac{\mathrm{d}y}{\mathrm{d}x}-(x+2)y=0.$$

Find another linearly independent solution to this equation.

Answer Substituting $1/x$ in to the equation gives $(x^2+x)(2/x^3)-(x^2-2)(-1/x^2)-(x+2)(1/x)=2x^{-1}+2x^{-2}+1-2x^{-2}-1-2x^{-1}=0$. Try $y=u/x$. We get

$$(x^2+x)\left[u''/x-2u'/x^2+2u/x^3\right]-(x^2-2)\left[u'/x-u/x^2\right]-(x+2)u/x=0.$$

This simplifies to

$$(x+1)u''=(x+2)u'$$

so that

$$\int\frac{\mathrm{d}u'}{u'}=\int\frac{x+2}{x+1}\,\mathrm{d}x$$

$$\ln|u'|=x+\ln|x+1|+C_1\qquad\text{or}\qquad u'=C_2(x+1)e^x$$

Therefore

$$u=C_2\int(x+1)e^x\,\mathrm{d}x=C_2xe^x+C_3.$$

So the general solution is $y=u/x=C_2e^x+C_3/x$, we recognize the second term as the solution $y_1=1/x$ we already know, so the other linearly independent solution can be chosen $y_2=e^x$.

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4. **Question** *Given that λ is a positive constant, find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ for the boundary value problem:*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\lambda y=0,\qquad y'(0)=0,\qquad y(3)=0.$$

Answer For $\lambda=\omega^2>0$, the general solution is $y=A\cos(\omega x)+B\sin(\omega x)$. From boundary conditions, $y'(0)=\omega B=0$ and $y(3)=A\cos(3\omega)+B\sin(3\omega)$. Obviously $B=0$ as $\omega>0$ by assumption. Then $A=0$ unless $\cos(3\omega)=0$, that is $\omega=\omega_n=(\pi/2+\pi n)/3$. The eigenvalues are therefore $\lambda=\lambda_n=\omega_n^2=(\pi/3)^2(n+1/2)^2$ and the eigenfunctions $\phi_n(x)=\cos((n+1/2)\pi x/3)$.

Question *Show that these eigenfunctions satisfy the orthogonality relation:*

$$\int_0^3\phi_n(x)\phi_m(x)\,\mathrm{d}x=0\qquad\textit{for}\qquad n\neq m.$$

Answer

$$\begin{aligned}\int_0^3\phi_n(x)\phi_m(x)\,\mathrm{d}x&=\int_0^3\cos[(n+1/2)\pi x/3]\cos[(m+1/2)\pi x/3]\,\mathrm{d}x\\&=\frac{1}{2}\int_0^3\left[\cos[(n-m)\pi x/3]+\cos[(n+m+1)\pi x/3]\right]\mathrm{d}x\\&=\frac{3}{2\pi}\left[\frac{\sin[(n-m)\pi x/3]}{n-m}+\frac{\sin[(n+m+1)\pi x/3]}{n+m+1}\right]_0^3=0\end{aligned}$$

as required, as $n\neq m$ by assumption and $n+m+1\neq 0$ since $n>0$ and $m>0$.

[9]

5. **Question** *Use a trial function of the form*

$$y=\sum_{n=0}^{\infty}a_nx^n$$

to find the solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x^2y=0.$$

Write the general solution in the form $y=Af(x)+Bxg(x)$, where $A=y(0)$ and $B=y'(0)$. Write down the first three non-zero terms of the expansions of $f(x)$ and $g(x)$.

Answer

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2a_2+6a_3x+12a_4x^2+20a_5x^3+\cdots+(n+2)(n+1)a_{n+2}x^n+\dots$$

We get a recurrence relation between a_{n+2} and a_{n-1} by equating coefficients of x^n. We find that

$$\frac{a_{n+2}}{a_{n-2}} = \frac{1}{(n+2)(n+1)}.$$

From the coefficients at x^0 through to x^3 we see that a_0 and a_1 are equal to A and B, and that $a_2 = a_3 = 0$. Then from the recurrence relation we have $a_4 = a_0/(3 \cdot 4) = A/12$, $a_5 = a_1/(4 \cdot 5) = B/20$, $a_6 = a_2/(5 \cdot 6) = 0$, $a_7 = a_3/(6 \cdot 7) = 0$, $a_8 = a_4/(7 \cdot 8) = A/672$ and $a_9 = a_5/(8 \cdot 9) = B/1440$. Thus $y = Af(x) + Bxg(x)$ where

$$f(x) = 1 + \frac{x^4}{12} + \frac{x^8}{672} + \dots$$

and

$$g(x) = 1 + \frac{x^4}{20} + \frac{x^8}{1440} + \dots$$

Question

Show that these solutions converge for all finite values of x.

Answer The ratio between successive nonzero terms in each of the two series is $a_{n+2}x^{n+2}/(a_{n-2}x^{n-2}) = x^4/((n+1)(n+2))$. The size of this ratio tends to zero for all finite values of x as n tends to ∞. The two series will therefore converge for all finite values of x.

[9]

6. **Question** *Explain what is meant by the terms* ordinary point, singular point *and* regular singular point *for the differential equation*

$$P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)y = 0,$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials.

Answer Any point where $P(x) = 0$ and so equation cannot be resolved for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is a singular point of the differential equation. Any point that is not singular is a regular point. If x_0 is a singular point and both limits

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} \qquad \text{and} \qquad \lim_{x \to x_0} (x - x_0)^2\frac{R(x)}{P(x)}$$

are finite, then x_0 is a regular singular point.

Question

Find the singular points of the differential equation

$$x^3(x^2 - 9)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x^2(x^2 - 9)\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + 1)y = 0$$

and for each singular point state whether it is regular or not.

Answer There are 3 singular points, $x = 0$, $x = 3$ and $x = -3$.

For the singular point $x = 0$ we have

$$\frac{xQ(x)}{P(x)} = \frac{x \cdot x^2(x^2 - 9)}{x^3(x^2 - 9)^2} \to \left[\frac{1}{x^2 - 9}\right]_0$$

finite as $x \to 0$, and

$$\frac{x^2R(x)}{P(x)} = \frac{x^2 \cdot (x^2 + 1)}{x^3(x^2 - 9)^2} \sim \frac{1}{81x} \to \infty$$

as $x \to 0$ so this is not a regular singular point.

For the singular point $x = 3$ we have

$$\frac{(x-3)Q(x)}{P(x)} = \frac{(x-3) \cdot x^2(x-3)(x+3)}{x^3(x-3)^2(x+3)^2} \to \left[\frac{1}{x(x+3)}\right]_{x=3}$$

finite, and

$$\frac{(x-3)^2R(x)}{P(x)} = \frac{(x-3)^2 \cdot (x^2+1)}{x^3(x-3)^2(x+3)^2} \to \left[\frac{x^2+1}{x^3(x+3)^2}\right]_{x=3}$$

also finite, so $x = 3$ is a regular singular point.

Calculations for $x = -3$ are identical to those for $x = 3$ with all $(x - 3)$ swapped with $(x + 3)$, and it comes out as a regular singular point, too

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$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 7e^{2t} \\ 7e^{2t} \end{bmatrix}.$$

Answer The characteristic equation of the matrix satisfy $(1-\lambda)^2 - 36 = 0$ so the eigenvalues are $\lambda = -5$ and $\lambda = 7$. The eigenvectors are $(3,-2)^T$ and $(3,2)^T$ respectively. The complementary function is therefore $C_1(3,-2)^T e^{-5t} + C_2(3,2)^T e^{7t}$, where C_1 and C_2 are constants.

The particular integral will be $\mathbf{w}\, e^{2t}$. Substituting into the differential equation and dividing through by e^{2t} gives:

$$2w_1 = 1w_1 + 9w_2 + 7 \qquad \text{and} \qquad 2w_2 = 4w_1 + w_2 + 7.$$

The solution is $w_1 = -2$ and $w_2 = -1$.

The complete solution is then

$$\mathbf{x} = C_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-5t} + C_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{7t} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t},$$

where C_1 and C_2 are arbitrary constants.

[8]

SECTION B

8. **Question** *Show that when $\lambda \leq 0$ the boundary value problem*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 10\frac{\mathrm{d}y}{\mathrm{d}x} + (25+\lambda)y = 0, \qquad y(0) = 0, \qquad y(\pi) = 0$$

has no eigenfunctions, but for appropriate values of $\lambda > 0$, the eigenfunctions are:

$$\phi_n(x) = e^{-5x}\sin(nx), \qquad n = 1, 2, 3 \cdots.$$

Answer Characteristic equation: $y = e^{mx}$ will be a solution if $m^2 + 10m + (25+\lambda) = 0$, that is for $m = -5 \pm \sqrt{-\lambda}$.

If $\lambda < 0$, both values of m will be real and distinct. Let $\lambda = -p^2$, $p > 0$, the general solution is then $y = Ae^{(-5+p)x} + Be^{(-5-p)x}$. The boundary conditions give $y(0) = 0 = A + B$ and $y(\pi) = 0 = (Ae^{\pi p} + Be^{-\pi p})e^{-5\pi}$. The only solution is $A = B = 0$ and there are no eigenfunctions.

For $\lambda = 0$ the general solution is $y = (A + Bx)e^{-5x}$. The boundary conditions give $y(0) = 0 = A$ and $y(\pi) = 0 = (A + B\pi)e^{-5\pi}$ and so $A = B = 0$ and there are no eigenfunctions.

For $\lambda = \omega^2$, $\omega > 0$, we have $y = e^{-5x}(A\cos(\omega x) + B\sin(\omega x))$.

The boundary conditions give $y(0) = 0 = A$ and then $y(\pi) = 0 = Be^{-5\pi}\sin(\omega\pi)$. Thus $B = 0$ unless $\sin(\omega\pi) = 0$. This will be so if $\omega = n$ for positive integer n, and the eigenfunctions are $e^{-5x}\sin(nx) = \phi_n(x)$ as requested.

Question

Further, show that $\int_0^\pi e^{10x}\phi_n(x)\phi_m(x)\,\mathrm{d}x = 0$ for $n \neq m$, and evaluate $\int_0^\pi e^{10x}\phi_n^2(x)\,\mathrm{d}x$.

Answer We have

$$\int_0^\pi e^{10x}\phi_n(x)\phi_m(x)\,\mathrm{d}x = \int_0^\pi e^{10x}e^{-5x}\sin(nx)e^{-5x}\sin(mx)\,\mathrm{d}x =$$

$$\int_0^\pi \sin(nx)\sin(mx)\,\mathrm{d}x = \frac{1}{2}\int_0^\pi [\cos((m-n)x) - \cos((m+n)x)]\,\mathrm{d}x.$$

For $m \neq n$, this integral evaluates to

$$\frac{1}{2}\left\{\left[\frac{\sin((m-n)x)}{m-n}\right]_0^\pi - \left[\frac{\sin((m+n)x)}{m+n}\right]_0^\pi\right\} = 0$$

as requested, as both denominators are nonzero since $m \neq n$ by assumption and both m and n are positive. .

For $m = n$, this integral becomes

$$\frac{1}{2}\int_0^\pi [1 - \cos(2nx)]\,\mathrm{d}x \qquad = \frac{1}{2}\left\{\pi - \left[\frac{\sin(2nx)}{2n}\right]_0^\pi\right\} = \frac{\pi}{2}.$$

[15]

$$y = \sum_{n=0}^{\infty} a_n x^{n+c}$$

to find two linearly independent solutions of the differential equation:

$$3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2-x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0. \qquad (1)$$

Write down the first three non-zero terms of each series.

Answer Substituting into the differential equation:

$$\begin{array}{llll}
-y = & -a_0x^c & -\cdots -a_nx^{c+n} & -\ldots \\
2\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2ca_0x^{c-1} & +2(c+1)a_1x^c & +\cdots +2(c+n+1)a_{n+1}x^{c+n} & +\ldots \\
-x\dfrac{\mathrm{d}y}{\mathrm{d}x} = & -ca_0x^c & -\cdots -(c+n)a_nx^{c+n} & -\ldots \\
3x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3c(c-1)a_0x^{c-1} & +3(c+1)ca_1x^c & +\cdots +3(c+n+1)(c+n)a_{n+1}x^{c+n} & +\ldots
\end{array}$$

The smallest power of x in this is x^{c-1}. Its coefficient is $(3c(c-1)+2c)a_0 = c(3c-1)a_0$. As a_0 is not zero by assumption, we have $c = 0$ or $c = 1/3$.

We get the recurrence relation by looking at the coefficient of x^{n+c}. This gives us

$$\frac{a_{n+1}}{a_n} = \frac{c+n+1}{(c+n+1)(3c+3n+2)} = \frac{1}{3c+3n+2}.$$

For $c = 0$ if we take $a_0 = 1$, we have one solution

$$y = 1 + \frac{1}{2}x + \frac{1}{2}\frac{1}{5}x^2 + \ldots,$$

and for $c = 1/3$ we have the other solution

$$y = x^{\frac{1}{3}}\left\{1 + \frac{1}{3}x + \frac{1}{3}\frac{1}{6}x^2 + \ldots\right\},$$

so we have two linearly independent solutions as requested.

Question

Show that both of these solutions converge for all values of $x > 0$.

Answer The ratio between successive terms in either of the series is $x/(3c+3n+2)$. This tends to zero as n tends to infinity for all finite values of x. The series therefore converges for all finite values of x.

Question

Write equation (1) in Sturm-Liouville form.

Answer To convert to the Sturm Liouville form, we can first divide through by $3x$ and then multiply through by a function $P(x)$ which changes the derivative part to the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) = P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + P'(x)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Such a function will satisfy the differential equation $P'(x) = [(2-x)/3x]P(x)$. Thus $\ln P = (2\ln x - x)/3$ so that $P = x^{2/3}e^{-x/3}$. The Sturm Liouville form is then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{2/3}e^{-x/3}\frac{\mathrm{d}y}{\mathrm{d}x}\right) - \frac{1}{3}x^{-1/3}e^{-x/3}y = 0.$$

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10. **Question** *Use a trial function of the form $y = \sum_{n=0}^{\infty} a_n x^n$ to find a series solution of the differential equation:*

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda y = 0. \qquad (2)$$

$$\frac{a_{n+2}}{a_n} = \frac{n(n+3)-\lambda}{(n+1)(n+2)}.$$

Answer

Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation gives

$$\begin{array}{rllllll}
\frac{d^2y}{dx^2} = 2a_2 & +2\cdot 3a_3x & +3\cdot 4a_4x^2 & +4\cdot 5a_5x^3 & +5\cdot 6a_6x^4 & +\cdots+(n+2)(n+1)a_{n-2}x^n & +\dots \\
-x^2\frac{d^2y}{dx^2} = & & -2a_2x^2 & -2\cdot 3a_3x^3 & -3\cdot 4a_4x^4 & -\cdots-n(n-1)a_nx^n & -\dots \\
-4x\frac{dy}{dx} = & -4a_1x & -4\cdot 2a_2x^2 & -4\cdot 3a_3x^3 & -4\cdot 4a_4x^4 & -\cdots-4na_nx^n & -\dots \\
\lambda y = \lambda a_0 & +\lambda a_1x & +\lambda a_2x^2 & +\lambda a_3x^3 & +\lambda a_4x^4 & +\cdots+\lambda a_nx^n & +\dots
\end{array}$$

The coefficient of x^n in the differential equation is

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 4na_n + \lambda a_n = 0$$

which gives the recurrence relation

$$\frac{a_{n+2}}{a_n} = \frac{n^2+3n-\lambda}{(n+2)(n+1)}$$

as requested.

Question

Show that the general solution to equation (2) is a linear combination of a series of odd powers of x and a series of even powers of x

Answer The solution is

$$y = a_0\left\{1 + \frac{a_2}{a_0}x^2 + \frac{a_2}{a_0}\frac{a_4}{a_2}x^4 + \dots\right\} + a_1\left\{x + \frac{a_3}{a_1}x^3 + \frac{a_3}{a_1}\frac{a_5}{a_3}x^5 + \dots\right\}$$

$$= a_0\left\{1 + \frac{-\lambda}{1\cdot 2}x^2 + \frac{-\lambda}{1\cdot 2}\frac{2\cdot 5-\lambda}{3\cdot 4}x^4 + \dots\right\} + a_1\left\{x + \frac{1\cdot 4-\lambda}{2\cdot 3}x^3 + \frac{1\cdot 4-\lambda}{2\cdot 3}\frac{3\cdot 6-\lambda}{4\cdot 5}x^5 + \dots\right\}$$

Question

Show that if $\lambda = m(m+3)$ and m is an even positive integer, the even series solution terminates and is just a polynomial, while if m is an odd positive integer, the series of odd powers of x terminates and becomes a polynomial. Write down the polynomials for the cases when $m = 1, 2, 3, 4$. Denote these polynomials by $P_m(x)$.

Answer If $\lambda = m(m+3)$ for some integer m, then the recurrent relationship for that m gives that $a_{m+2}/a_m = 0$, hence $a_{m+2} = 0$ and all subsequent coefficients of the same parity vanish, too, and this series terminates. If m is odd, it is the series of odd powers which terminates, while if m is even, the series of even powers terminates.

If $m = 1$, $\lambda = 1\cdot 4 = 4$, $a_3 = 0$, so $P_1(x) = x$.

If $m = 2$, $\lambda = 2\cdot 5 = 10$, $a_4 = 0$, $a_2/a_0 = -10/(1\cdot 2)$, so $P_2(x) = 1 - 5x^2$.

If $m = 3$, $\lambda = 3\cdot 6 = 18$, $a_5 = 0$, $a_3/a_1 = (4-18)/(2\cdot 3) = -7/3$, so $P_3(x) = x - \frac{7}{3}x^3$.

Question *Show that the Sturm-Liouville form of equation (2) is*

$$\frac{d}{dx}\left((1-x^2)^2\frac{dy}{dx}\right) + (1-x^2)\lambda y = 0.$$

Answer Differentiation using the product and chain rules in the above equation gives

$$(1-x^2)^2\frac{d^2y}{dx^2} + 2(1-x^2)(-2x)\frac{dy}{dx} + (1-x^2)\lambda y = 0.$$

Dividing through by $(1-x^2)$ gives equation (2), which proves the required.

Question

$$\int_{-1}^{1} (1-x^2) P_n(x) P_m(x)\,\mathrm{d}x = 0.$$

Answer The integration is from -1 to 1. This means that the integral of an odd function of x is zero. Therefore the integral of an odd order P with an even order P is zero. We thus only have to evaluate the integral with P_1 and P_3. It is

$$\int_{-1}^{1} (1-x^2)x\left(x-\frac{7}{3}x^3\right)\mathrm{d}x = 2\int_0^1 \left(x^2 - \frac{10}{3}x^4 + \frac{7}{3}x^6\right)\mathrm{d}x$$
$$= 2\left[\frac{x^3}{3} - \frac{10}{3}\frac{x^5}{5} + \frac{7}{3}\frac{x^7}{7}\right]_0^1 = 2\left(\frac{1}{3} - \frac{2}{3} + \frac{1}{3}\right) = 0.$$

as required.

[15]

11. **Question** *Show that eigenvalue* $\lambda = -2$ *is a double root of the characteristic equation of the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}.$$

and find the other eigenvalue.

Answer The characteristic determinant is

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = -(\lambda^3 - 12\lambda - 16).$$

If $\lambda_1 = \lambda_2 = -2$ is a double root then we should have $\lambda^3 - 12\lambda - 16 = (\lambda+2)^2(\lambda-\lambda_3)$. By multiplying through and equating coefficients at powers of λ, we get $4-\lambda_3 = 0$, $4-4\lambda_3 = -12$ and $4\lambda_3 = 4$. All these three equations can, indeed, be satisfied (which proves that $\lambda = -2$ is indeed a double root) by $\lambda_3 = 4$.

Question *Show that the vectors* $(1,1,0)^T$ *and* $(1,0,-1)^T$ *are eigenvectors of* $\mathbf{A}$ *and find the third eigenvector, writing it in the form* $(1, u_2, u_3)^T$.

Answer Multiplication gives $\mathbf{A}(1,1,0)^T = (-2,-2,0)^T = -2(1,1,0)^T$, so $(1,1,0)^T$ is an eigenvector $\mathbf{e}_1$ with eigenvalue $\lambda_1 = -2$. Similarly, $\mathbf{A}(1,0,-1)^T = (-2,0,2)^T = -2(1,0,-1)^T$, so $(1,0,-1)^T$ is an eigenvector $\mathbf{e}_2$ with the same eigenvalue $\lambda_2 = -2$.

The third eigenvector can be found by pre-multiplying the first column of $\mathbf{A} - \lambda_2\mathbf{I}$ by the matrix $\mathbf{A} - \lambda_1\mathbf{I}$. That is

$$\mathbf{e}_3 = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \\ 36 \end{bmatrix}.$$

We can take a factor of 18 out so that $\mathbf{e}_3 = (1,1,2)^T$. Its eigenvalue is $\lambda_3 = 4$.

Question *Find a matrix* $\mathbf{P}$ *and a diagonal matrix* $\mathbf{D}$ *such that*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}.$$

Answer The matrix $\mathbf{P}$ is the matrix whose columns are the eigenvectors, so $\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$ and $\mathbf{D}$ is a diagonal matrix whose elements are the eigenvalues, $\mathbf{D} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

Question *Transform the set of differential equations:*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{f}(t),$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^T$, *and* $\mathbf{f}(t)$ *is a given vector-function of time, into the form:*

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{D}\mathbf{y} + \mathbf{c}(t),$$

terms of the components of $\mathbf{f}(t)$.

Answer The required transformation is $\mathbf{x} = \mathbf{P}\mathbf{y}$ which gives $\dot{\mathbf{x}} = \mathbf{P}\dot{\mathbf{y}} = \mathbf{APy} + \mathbf{f}(t)$. Then $\mathbf{P}^{-1}\mathbf{P}\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{APy} + \mathbf{P}^{-1}\mathbf{f}(t)$, that is $\dot{\mathbf{y}} = \mathbf{Dy} + \mathbf{c}(t)$, where $\mathbf{c}(t) = \mathbf{P}^{-1}\mathbf{f}(t)$. The inverse matrix is

$$P^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 & -1 \\ 2 & -2 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

Therefore $c_1 = (-f_1 + 3f_2 - f_3)/2$, $c_2 = f_1 - f_2$ and $c_3 = (f_1 - f_2 + f_3)/2$.

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12. **Question** *Show that* $\mathbf{x} = (1,1)^T e^{3t}$ *is one solution of*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\mathbf{x}.$$

Answer If we substitute given $\mathbf{x}$ into the equation, we get $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t}$ for the left-hand side, and $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t}$ for the right-hand side, so the equation is satisifed.

Question *Find a second solution and hence write down the general solution.*

Answer We find the eigenvalues by solving $\begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0.$ This has a double root $\lambda = 3$.

There is no second eigenvector for this eigenvalue, hence we look for a second solution $(1,1)^T te^{3t} + \mathbf{w}e^{3t}$. We get

$$\begin{bmatrix} 3t + 1 + 3w_1 \\ 3t + 1 + 3w_2 \end{bmatrix} e^{3t} = \begin{bmatrix} 3t + 2w_1 + w_2 \\ 3t - w_1 + 4w_2 \end{bmatrix} e^{3t}.$$

Both components lead to the same equation, $w_1 - w_2 = -1$. We can take $w_1 = 0$ and $w_2 = 1$. The second solution is then $\mathbf{x} = (1,1)^T te^{3t} + (0,1)^T e^{3t}$. The general solution is then $\mathbf{x} = A(1,1)^T e^{3t} + B[(1,1)^T te^{3t} + (0,1)^T e^{3t}]$.

Question

Find a linear transformation, $\mathbf{x} = \mathbf{Py}$, *which will decouple the differential equations represented in the matrix form as*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\mathbf{x} + \mathbf{f}(t),$$

where $\mathbf{f}(t)$ *is some known vector-function of* t, *and write down the decoupled differential equations. Solve these differential equations and hence determine* $\mathbf{x}(t)$ *when* $\mathbf{f}(t) = (0,1)^T e^{3t}$.

Answer We take the columns of $\mathbf{P}$ equal to the eigenvector and the generalized eigenvector, that is the first column $(1,1)^T$ and second column $(0,1)^T$. Thus

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Substitution $\mathbf{x} = \mathbf{Py}$ gives

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{P}^{-1}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\mathbf{f} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\mathbf{f}.$$

By components, we have two decoupled equations: $\dot{y}_1 = 3y_1 + y_2 + f_1$ and $\dot{y}_2 = 3y_2 - f_1 + f_2$ — the equation for y_2 does not involve y_1 (but not vice versa!).

We solve equation for y_2 first: $\dot{y}_2 = 3y_2 + e^{3t}$. This is a first order linear equation whose integrating factor is e^{-3t}. The solution is $y_2 = (A + t)e^{3t}$.

The equation for y_1 is then $\dot{y}_1 = 3y_1 + (A + t)e^{3t}$. This is also a first order linear equation with integrating factor e^{-3t}. The solution is $y_1 = (t^2/2 + At + B)e^{3t}$. Then $x_1 = y_1 = (t^2/2 + At + B)e^{3t}$, and $x_2 = y_1 + y_2 = (t^2/2 + (A+1)t + A + B)e^{3t}$.

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