JANUARY 2007 EXAMINATIONS

Bachelor of Arts : Year 2
Bachelor of Science : Year 1
Bachelor of Science : Year 2
Bachelor of Science : Year 3
Master of Chemistry : Year 2
Master of Mathematics : Year 2
Master of Physics : Year 4
No qualification aimed for : Year 1

ORDINARY DIFFERENTIAL EQUATIONS

TIME ALLOWED: TWO HOURS AND A HALF

INSTRUCTIONS TO CANDIDATES

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

SECTION A

1. Find the general solutions of the differential equations:

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + (1+x)y^2 = 2x^3y^2,$$

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + (1+x^2)y = 2x^3.$$

[7 marks]

2. Solve the initial value problem:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 11\frac{\mathrm{d}y}{\mathrm{d}x} + 28y = 28x^2 + 22x + 30, \qquad y(0) = 0, \qquad y'(0) = 1.$$

[7 marks]

3. Show that $y = 1/x$ is a solution of the differential equation

$$(x^2+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (x^2-2)\frac{\mathrm{d}y}{\mathrm{d}x} - (x+2)y = 0.$$

Find another linearly independent solution to this equation.

[8 marks]

4. Given that λ is a positive constant, find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ for the boundary value problem:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \lambda y = 0, \qquad y'(0) = 0, \qquad y(3) = 0.$$

Show that these eigenfunctions satisfy the orthogonality relation:

$$\int_0^3 \phi_n(x)\phi_m(x)\,\mathrm{d}x = 0 \qquad \text{for} \qquad n \neq m.$$

[9 marks]

5. Use a trial function of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to find the solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x^2 y = 0.$$

Write the general solution in the form $y = Af(x) + Bxg(x)$, where $A = y(0)$ and $B = y'(0)$. Write down the first three non-zero terms of the expansions of $f(x)$ and $g(x)$.

Show that these solutions converge for all finite values of x.

[9 marks]

6. Explain what is meant by the terms *ordinary point*, *singular point* and *regular singular point* for the differential equation

$$P(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)y = 0,$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials.

Find the singular points of the differential equation

$$x^3(x^2-9)^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x^2(x^2-9)\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2+1)y = 0$$

and for each singular point state whether it is regular or not.

[7 marks]

7. Find the general solution of the differential equations:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 7e^{2t} \\ 7e^{2t} \end{bmatrix}.$$

[8 marks]

SECTION B

8. Show that when $\lambda \leq 0$ the boundary value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 10\frac{\mathrm{d}y}{\mathrm{d}x} + (25+\lambda)y = 0, \qquad y(0) = 0, \qquad y(\pi) = 0$$

has no eigenfunctions, but for appropriate values of $\lambda > 0$, the eigenfunctions are:

$$\phi_n(x) = e^{-5x}\sin(nx), \qquad n = 1, 2, 3\cdots.$$

Further, show that $\int_0^\pi e^{10x}\phi_n(x)\phi_m(x)\,\mathrm{d}x = 0$ for $n \neq m$, and evaluate $\int_0^\pi e^{10x}\phi_n^2(x)\,\mathrm{d}x$.

[15 marks]

9. Use a trial function of the form:

$$y = \sum_{n=0}^{\infty} a_n x^{n+c}$$

to find two linearly independent solutions of the differential equation:

$$3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2-x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0. \qquad (1)$$

Write down the first three non-zero terms of each series.

Show that both of these solutions converge for all values of $x > 0$.

Write equation (1) in Sturm-Liouville form.

[15 marks]

10. Use a trial function of the form $y = \sum_{n=0}^{\infty} a_n x^n$ to find a series solution of the differential equation:

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda y = 0. \tag{2}$$

Show that the recurrence relation between the coefficients a_{n+2} and a_n is

$$\frac{a_{n+2}}{a_n} = \frac{n(n+3)-\lambda}{(n+1)(n+2)}.$$

Show that the general solution to equation (2) is a linear combination of a series of odd powers of x and a series of even powers of x

Show that if $\lambda = m(m+3)$ and m is an even positive integer, the even series solution terminates and is just a polynomial, while if m is an odd positive integer, the series of odd powers of x terminates and becomes a polynomial. Write down the polynomials for the cases when $m = 1, 2, 3, 4$. Denote these polynomials by $P_m(x)$. Show that the Sturm-Liouville form of equation (2) is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)^2\frac{\mathrm{d}y}{\mathrm{d}x}\right) + (1-x^2)\lambda y = 0.$$

Show that for all $m, n = 1, 2, 3$, where $m \neq n$,

$$\int_{-1}^{1} (1-x^2)P_n(x)P_m(x)\,\mathrm{d}x = 0.$$

[15 marks]

11. Show that eigenvalue $\lambda = -2$ is a double root of the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}.$$

and find the other eigenvalue. Show that the vectors $(1, 1, 0)^T$ and $(1, 0, -1)^T$ are eigenvectors of $\mathbf{A}$ and find the third eigenvector, writing it in the form $(1, u_2, u_3)^T$. Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}.$$

Transform the set of differential equations:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{f}(t),$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^T$, and $\mathbf{f}(t)$ is a given vector-function of time, into the form:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{D}\mathbf{y} + \mathbf{c}(t),$$

where $\mathbf{A}$ and $\mathbf{D}$ are the matrices given/obtained above. Write down expressions for the components of $\mathbf{c}(t)$ in terms of the components of $\mathbf{f}(t)$.

[15 marks]

12. Show that $\mathbf{x} = (1, 1)^T e^{3t}$ is one solution of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \mathbf{x}.$$

Find a second solution and hence write down the general solution.

Find a linear transformation, $\mathbf{x} = \mathbf{P}\mathbf{y}$, which will decouple the differential equations represented in the matrix form as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \mathbf{x} + \mathbf{f}(t),$$

where $\mathbf{f}(t)$ is some known vector-function of t, and write down the decoupled differential equations. Solve these differential equations and hence determine $\mathbf{x}(t)$ when $\mathbf{f}(t) = (0, 1)^T e^{3t}$.

[15 marks]