Solutions to MATH 201 JAN 2006

The solutions are all similar to questions set for homework except where marked bw for bookwork.

SECTION A

1.
$$\frac{dy}{dx} = \frac{x^2 - x - 1}{x} y^2.$$

This is separable. Separating the variables gives

$$\int \frac{dy}{y^2} = \int (x - 1 - 1/x)dx = -\frac{1}{y} = x^2/2 - x - \ln|x| + C.$$

So that
$$y = -\frac{1}{x^2/2 - x - \ln|x| + C}$$

The second equation is linear

$$\frac{dy}{dx} + \frac{2}{1+x}y = x - 1 \qquad \text{has integrating factor} \qquad \exp\Big(\int \frac{2dx}{1+x}\Big) = \exp(2\ln|1+x|) = (1+x)^2.$$

Multiplying through by the integrating factor gives

$$(1+x)^2\frac{dy}{dx} + 2(1+x)y = \frac{d}{dx}\big((1+x)^2y\big) = x^3 + x^2 - x - 1$$

Integrating this equation gives

$$(1+x)^2 y = \frac{x^3}{3} + \frac{x^4}{4} - x - \frac{x^2}{2} + C$$

2. The solution $y = e^{mx}$ will satisfy the homogeneous equation if $m^2 + 13m + 40 = 0$. This has roots $m = -5$ and $m = -8$ so the Complementary Function is $Ae^{-5x} + Be^{-8x}$.

To find the particular integral we try $y = \alpha x^2 + \beta x + \gamma$. We find

$$2\alpha + 13(2\alpha x + \beta) + 40(\alpha x^2 + \beta x + \gamma) = 40x^2 + 146x + 241$$

This is satisfied if $\alpha = 1$, $26\alpha + 40\beta = 146$, so that $\beta = 3$ and $2\alpha + 13\beta + 40\gamma = 241$, so that $\gamma = 5$.

The general solution is then

$$y = Ae^{-5x} + Be^{-8x} + x^2 + 3x + 5$$

From the initial conditions we get

$$y(0) = A + B + 5 = 10, \qquad y'(0) = -31 = -5A - 8B + 3$$

Therefore $A = 2$ and $B = 3$.

The solution is therefore $y = 2e^{-5x} + 3e^{-8x} + x^2 + 3x + 5$

3. Substituting x in to the equation gives $-4x+4x=0$
Try $y = xu$. We get

$$(1+x^2)\big[xu''+2u'\big]-4x\big[xu'+u\big]+4xu=0$$

This simplifies to

$$u''=\left(\frac{-2}{x}+\frac{4x}{1+x^2}\right)u' \qquad \text{so that} \qquad \int\frac{d(u')}{u'}=-2\int\frac{dx}{x}+\int\frac{4x\,dx}{x^2+1}$$

$$\ln(u')=-2\ln x+2\ln(x^2+1) \qquad \text{or} \qquad u'=\frac{(1+x^2)^2}{x^2}=\frac{1}{x^2}+2+x^2$$

Therefore

$$u=-\frac{1}{x}+2x+x^3/3$$

The second solution is therefore $y=\frac{x^4}{3}+2x^2-1$.

4. For $\lambda=\omega^2$, The general solution is $y=A\cos(\omega x)+B\sin(\omega x)$. $y(0)=A=0$ and $y'(\pi)=B\omega\cos(\omega\pi)$. $B=0$ unless $\cos(\omega\pi)=0$ that is if $\omega=(n+1/2)$. The eigenvalues are therefore $\lambda=(n+1/2)^2$ and the eigenfunctions $\sin((n+1/2)x)$.

$$\int_0^\pi\phi_n(x)\phi_m(x)dx=\int_0^\pi\sin[(n+1/2)x]\sin[(m+1/2)x]dx$$

$$=\frac{1}{2}\int_0^\pi\big[\cos[(n-m)x]-\cos[(n+m+1)x]\big]dx=\frac{1}{2}\left[\frac{\sin[(n-m)x]}{n-m}-\frac{\sin[(n+m+1)x]}{n+m+1}\right]_0^\pi=0$$

5.

$$\frac{d^2y}{dx^2}=2a_2+6a_3x+12a_4x^2+20a_5x^3+...+(n+2)(n+1)a_{n+2}x^n+...$$

$$2xy=\quad 2xa_0+2a_1x^2+2a_2x^3+2a_3x^4+....+2a_{n-1}x^n+2a_nx^{n+1}...$$

We get a recurrence relation between a_{n+2} and a_{n-1} by equating coefficients of x^n. We find that

$$\frac{a_{n+2}}{a_{n-1}}=\frac{-2}{(n+2)(n+1)}.$$

We see that a_0 and a_1 are equal to A and B, that $a_2=0$, $a_3=-a_0/3$, $a_4=-2a_1/12$, $a_5=0$. Thus

$$f(x)=1-\frac{2}{6}x^3+\frac{2}{6.5}\frac{2}{6}x^6+...$$

and

$$g(x)=1-\frac{2}{4.3}x^3+\frac{2}{7.6}\frac{2}{4.3}x^6+...$$

The ratio between successive terms in each of the two series is $-2x^3/((n+3)(n+2))$. The size of this ratio tends to zero for all finite values of x as n tends to ∞. The two series will therefore converge for all finite values of x.

6. Any point where $P(x) = 0$ and so d^2y/dx^2 is not defined is a singular point of the differential equation. All other points are ordinary points. If x_0 is a singular point and

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} \quad \text{exists} \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} \quad \text{exists}$$

the singular point is a regular singular point. If either limit does not exist, the point $x = x_0$ is an irregular singular point.

There are 3 singular points, $x = 0$, $x = 1$ and $x = 2$.

For $x = 0$ we have $(x + 4)/(x^2 - 3x + 2)$ is finite, but $5/[x(x^2 - 3x + 2)]$ is not as $x \to 0$, so $x = 0$ is an irregular singular point.

For $x = 1$, $(x + 4)/[x(x - 2)]$ and $5(x - 1)/[x^3(x - 2)]$ are both finite as $x \to 1$, so this point is a regular singular point

For $x = 2$, we have $(x + 4)/[x(x - 1)$ and $5(x - 2)/[x^3(x - 1)]$ are both finite as $x \to 2$ so this point is also a regular singular point.

7. The eigenvalues of A satisfy $(2 - \lambda)(6 - \lambda) + 3 = 0$ so the eigenvalues are $\lambda = 3$ and $\lambda = 5$. The eigenvectors are $(3, -1)^T$ and $(1, -1)^T$. The complementary function is therefore $A_1(3, -1)^T e^{3t} + A_2(1, -1)^T e^{5t}$, where A_1 and A_2 are constants.

The particular integral will be $\mathbf{w}e^{2t}$. Substituting into the differential equation and dividing through by e^{2t} gives:

$$2w_1 = 2w_1 - 3w_2 - 6 \qquad \text{and} \qquad 2w_2 = w_1 + 6w_2 - 9.$$

The solution is $w_1 = 17$ and $w_2 = -2$.

The complete solution is then

$$\mathbf{x} = A_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{3t} + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{5t} + \begin{pmatrix} 17 \\ -2 \end{pmatrix} e^{2t},$$

where A_1 and A_2 are constants

SECTION B

8. $y = e^{mx}$ will be a solution if $m^2 + 4m + \lambda = 0$, that is for $m = -2 \pm \sqrt{4 - \lambda}$.

If $\lambda < 4$, both values of m will be real and distinct. Let $\lambda = 4 - p^2$, the general solution is then $y = Ae^{(p-2)x} + Be^{-(p+2)x}$. The boundary conditions give $y(0) = 0 = A + B$ and $y'(1) = 0 = (p - 2)Ae^{p-2} - (p + 2)Be^{-p-2}$. The only solution is $A = B = 0$.

For $\lambda = 0$ the general solution is $y = (A + Bx)e^{-2x}$. The boundary conditions give $y(0) = 0 = A$ and $y'(1) = 0 = (B - 2A - 2B)e^{-2}$ and so $A = B = 0$.

For $\lambda = 4 + \omega^2$, we have $y = e^{-2x}\Big(A\cos(\omega x) + B\sin(\omega x)\Big)$.

The boundary conditions give $y(0) = 0 = A$ and then $y'(1) = 0 = Be^{-2}(-2\sin\omega + \omega\cos\omega)$. Thus $B = 0$ unless $-2\sin(\omega) + \omega\cos\omega = 0$. This will be so if $\omega = 2\tan\omega$. eigenfunctions $e^{-2x}\sin(\omega_n x)$. To convert to the Sturm Liouville form, we have to multiply through by a function $P(x)$ which changes the derivative part to the form

$$\frac{d}{dx}\left(P(x)\frac{dy}{dx}\right) = P(x)\frac{d^2y}{dx^2} + P'(x)\frac{dy}{dx}.$$

Such a function will satisfy the differential equation $P'(x) = 4P(x)$. This function is e^{4x}. The Sturm Liouville form is therefore

$$\frac{d}{dx}\left(e^{4x}\frac{dy}{dx}\right) + \lambda e^{4x} y = 0.$$

If $\phi_n(x)$ is the n'th eigenfunction, with eigenvalue λ_n,

$$\frac{d}{dx}\left(e^{4x}\frac{d\phi_n}{dx}\right) = -\lambda_n e^{4x}\phi_n.$$

Multiply this by a different eigenfunction $\phi_m(x)$ and integrate from 0 to 1

$$\int_0^1 \phi_m \frac{d}{dx}\left(e^{4x}\frac{d\phi_n}{dx}\right)dx = -\lambda_n \int_0^1 e^{4x}\phi_n\phi_m dx = \left[\phi_m(x)e^{4x}\frac{d}{dx}\phi_n(x)\right]_0^1 - \int_0^1 e^{4x}\frac{d\phi_m}{dx}.\frac{d\phi_n}{dx}.$$

The integrated term is zero because at $x = 0$, $\phi_m(0) = 0$ and at $x = 1$, $\phi_n'(1) = 0$. If we now write the equation for $\phi_m(x)$, multiply by $\phi_n(x)$ and integrate from 0 to 1, we see that if we integrate by parts, the integrated term vanishes leaving us with the same integral of ϕ_n' and ϕ_m'. We see then that

$$-\lambda_n \int_0^1 e^{4x}\phi_n\phi_m dx = -\lambda_m \int_0^1 e^{4x}\phi_n\phi_m dx.$$

For this to be true with $\lambda_n \neq \lambda_m$, $\int_0^1 e^{4x}\phi_n\phi_m dx = 0$.

9. Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation gives

λy	$=$	λa_0	$+\lambda a_1 x$	$+\lambda a_2 x^2$	$+\lambda a_3 x^3$	$+\lambda a_4 x^4$	$+...$	$+\lambda a_n x^n$	$+...$
$\frac{d^2y}{dx^2}$	$=$	$2a_2$	$+2.3a_3 x$	$+3.4a_4 x^2$	$+4.5a_5 x^3$	$+5.6a_6 x^4$	$+...$	$+(n+2)(n+1)x^n$	$+...$
$-x^2\frac{d^2y}{dx^2}$	$=$			$-2a_2 x^2$	$-2.3a_3 x^3$	$-3.4a_4 x^4$	$-...$	$-n(n-1)a_{n+2}x^n$	$-...$

The coefficient of x^n in the differential equation is

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n + \lambda a_n = 0$$

This gives the recurrence relation

$$\frac{a_{n+2}}{a_n} = \frac{n(n-1) - \lambda}{(n+2)(n+1)}.$$

The solution is

$$a_0\left\{1 + \frac{a_2}{a_0}x^2 + \frac{a_2}{a_0}\frac{a_4}{a_2}x^4 + ...\right\} + a_1\left\{x + \frac{a_3}{a_1}x^3 + \frac{a_3}{a_1}\frac{a_5}{a_3}x^5 + ...\right\}$$

$$= a_0\left\{1 + \frac{-\lambda}{1.2}x^2 + \frac{-\lambda}{1.2}\frac{2-\lambda}{3.4}x^4 + ...\right\} + a_1\left\{x + \frac{-\lambda}{2.3}x^3 + \frac{-\lambda}{2.3}\frac{3.2-\lambda}{4.5}x^5 + ...\right\}$$

Clearly if $\lambda = m(m-1)$ for some integer m, a_{m+2}/a_m=0, and this series terminates. If m is odd, it is the series of odd powers which terminates, while if m is even, the series of even powers terminates.

If $m = 2$, $\lambda = 2$, $a_4 = 0$, $a_2/a_0 = -1$, so $Q_2(x) = 1 - x^2$.

If $m = 3$, $\lambda = 6$, $a_5 = 0$, $a_3/a_1 = -1$, so $Q_3(x) = x - x^3$.
If $m = 4$, $\lambda = 12$, $a_6 = 0$, so $Q_4(x) = 1 - 6x^2 + 5x^4$.
If $m = 5$, $\lambda = 20$, $a_7 = 0$, so $Q_5(x) = x - 10x^3/3 + 7x^5/3$.

The integration is from -1 to 1. This means that the integral of an odd function of x is zero. Therefore the integral of an odd order Q with an even order Q is zero. We thus only have to evaluate the integral with Q_2 and Q_4 and the integral withQ_3 and Q_5. These are

$$\int_{-1}^{1} \frac{(1-x^2)(1-6x^2+5x^4)}{1-x^2} dx = \left[x - 2x^3 + x^5\right]_{-1}^{1} = 0$$

and

$$\int_{-1}^{1} \frac{(x-x^3)(x-10x^3/3+7x^5/3)}{1-x^2} dx = \left[\frac{x^3}{3} - \frac{2x^5}{3} + x^7/3\right]_{-1}^{1} = 0.$$

10. Substituting into the differential equation:

$$\begin{array}{lllllll}
-y & = & & -a_0 x^c & - \ldots & -a_n x^{c+n} & - \ldots \\
3\frac{dy}{dx} & = & 3ca_0 x^{c-1} & +3(c+1)a_1 x^c & + \ldots & +3(c+n)a_{n+1}x^{c+n} & + \ldots \\
-x\frac{dy}{dx} & = & & ca_0 x^c & - \ldots & -(c+n)a_n x^{c+n} & - \ldots \\
4x\frac{d^2y}{dx^2} & = & 4c(c-1)a_0 x^{c-1} & +4(c+1)ca_1 x^c & + \ldots & +4(c+n+1)(c+n)a_{n+1}x^{c+n} & + \ldots
\end{array}$$

The smallest power of x in this is x^{c-1}. Its coefficient is $(4c(c-1) + 3c)a_0$. a_0 is not zero by hypothesis and so $c = 0$ or $c = 1/4$.

We get the recurrence relation by looking at the coefficient of x^{n+c}. This gives us

$$\frac{a_{n+1}}{a_n} = \frac{c+n+1}{(c+n+1)(4c+4n+3)} = \frac{1}{4c+4n+3}$$

For $c = 0$ if we take $a_0 = 1$,

$$y = 1 + \frac{1}{3}x + \frac{1}{3}\frac{1}{7}x^2 + \ldots$$

and for $c = 1/4$ we have

$$y = x^{\frac{1}{4}}\left\{1 + \frac{1}{4}x + \frac{1}{4}\frac{1}{8}x^2 + \ldots\right\}$$

The ratio between successive terms in either of the series is $x/(4c+4n+3)$. This tends to zero as n tends to infinity for all finite values of x. The series therefore converges for all finite values of x.

To convert to the Sturm Liouville form, we have to multiply through by a function $P(x)$ which changes the derivative part to the form

$$\frac{d}{dx}\left(P(x)\frac{dy}{dx}\right) = P(x)\frac{d^2y}{dx^2} + P'(x)\frac{dy}{dx}.$$

Such a function will satisfy the differential equation $P'(x) = [(3-x)/4x]P(x)$.

Thus $\ln P = (3\ln x - x)/4$ so that $P = x^{3/4}e^{-x/4}$. The Sturm Liouville form is then

$$\frac{d}{dx}\left(x^{3/4}e^{-x/4}\frac{dy}{dx}\right\} - \frac{x^{-1/4}}{4}e^{-x/4}y = 0.$$

11. $A(1,2,-1)^T = (1,2,-1)^T$, so $(1,2,-1)^T$ is an eigenvector with eigenvalue $\lambda = 1$. Alternatively, we can find the other eigenvalue from the determinant

$$\det\begin{vmatrix} 7-\lambda & 2 & 10 \\ -8 & -3-\lambda & -16 \\ 5 & 1 & 4-\lambda \end{vmatrix} = 0 = -\lambda^3 + 8\lambda^2 + 17\lambda + 10 = -(\lambda-1)(\lambda-2)(\lambda-5).$$

The roots are $\lambda = 1$, $\lambda = 2$ and $\lambda = 5$.

The third eigenvector can be found by multiplying the first column of $A - \lambda_2 I$ by the matrix $A - \lambda_1 I$. That is

$$\mathbf{e}_3 = \begin{pmatrix} 6 & 2 & 10 \\ -8 & -4 & -16 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ -8 \\ 1 \end{pmatrix} = \begin{pmatrix} 24 \\ -24 \\ 0 \end{pmatrix}.$$

We can take a factor of 24 out so that $\mathbf{e}_3 = (1,-1,0)^T$. Its eigenvalue is 5.

The second eigenvector can be found by multiplying the first column of $A-\lambda_3 I$ by the matrix $A - \lambda_1 I$. That is

$$\mathbf{e}_3 = \begin{pmatrix} 6 & 2 & 10 \\ -8 & -4 & -16 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -8 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ -3 \end{pmatrix}.$$

We can take a factor of 3 out so that $\mathbf{e}_3 = (2,0,-1)^T$. Its eigenvalue is 2.

The second eigenvector can be found by taking the cofactors of the elements of the first row of $A - 2I$. This gives a multiple of $(2,0,-1)^T$.

The third eigenvector can be found by taking the cofactors of the elements of the first row of $A - 5I$. This gives a multiple of $(1,-1,0)^T$.

The matrix P is the matrix whose columns are the eigenvectors, so

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Write $\mathbf{x} = P\mathbf{y}$ and then $\dot{\mathbf{x}} = P\dot{\mathbf{y}} = AP\mathbf{y} + \mathbf{f}(t)$.

Then $P^{-1}P\dot{\mathbf{y}} = P^{-1}AP\mathbf{y} + P^{-1}\mathbf{f}(t)$
or $\dot{\mathbf{y}} = D\mathbf{y} + \mathbf{c}(t)$, where $\mathbf{c}(t) = P^{-1}\mathbf{f}(t)$. The inverse matrix P^{-1} is

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & -3 \\ 2 & 1 & 4 \end{pmatrix}$$

Therefore $c_1 = f_1 + f_2 + 2f_3$, $c_2 = -f_1 - f_2 - 3f_3/$ and $c_3 = +2f_1 + f_2 + 4f_3$.

12.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} e^{5t} = \begin{pmatrix} 2 & 1 \\ -9 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{5t}.$$

We find the eigenvalues by solving

$$\det\begin{vmatrix} 2-\lambda & 1 \\ -9 & 8-\lambda \end{vmatrix} = 0$$

This has a double root $\lambda = 5$

We look for a second solution $(1,3)^T te^{5t} + \mathbf{w}e^{5t}$. We get

$$\begin{pmatrix} 5t+1+5w_1 \\ 15t+3+5w_2 \end{pmatrix} e^{5t} = \begin{pmatrix} 5t+2w_1+w_2 \\ 15t-9w_1+8w_2 \end{pmatrix} e^{5t}.$$

Therefore

$$3w_1 - w_2 = -1 \qquad \text{and} \qquad 9w_1 - 3w_2 = -3.$$

We can take $w_2 = 1$ and $w_1 = 0$. The second solution is then $y = (1,3)^T te^{5t} + (0,1)^T e^{5t}$ The general solution is then $y = A(1,3)^T e^{5t} + B[(1,3)^T te^{5t} + (0,1)^T e^{5t}]$

We take P to have the first column $(1,3)^T$ and second column $(0,1)^T$ Thus

$$P = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \qquad \text{and} \qquad P^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

Writing $\mathbf{x} = P\mathbf{y}$

$$\frac{d\mathbf{y}}{dt} = P^{-1}\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -9 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \mathbf{f} = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \mathbf{f}$$

The equations are now decoupled. The equation for y_2 does not involve y_1.

We have $\dot{y}_1 = 5y_1 + y_2 + f_1$ and $\dot{y}_2 = 5y_2 - 3f_1 + f_2$

We have to solve $\dot{y}_2 = 5y_2 + e^{5t}$. This is a first order linear equation whose integrating factor is e^{-5t}. The solution is $y_2 = (A+t)e^{5t}$.

The equation for y_1 is $\dot{y}_1 = 5y_1 + (A+t)e^{5t}$. This is also a first order linear equation with integrating factor e^{-5t}. The solution is $y_1 = (B + At + t^2/2)e^{5t}$. Then

$x_1 = y_1 = (B + At + t^2/2)e^{5t}$, and $x_2 = 3y_1 + y_2 = (A + 3B + (3A+1)t + 3t^2/2)e^{5t}$.