MATH 201 Jan 2006

ORDINARY DIFFERENTIAL EQUATIONS

TIME ALLOWED: TWO HOURS AND A HALF

Instructions to candidates

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

SECTION A

1. Find the general solutions for the differential equations:

$$x\frac{dy}{dx} + (1+x)y^2 = x^2y^2,$$

$$(1+x)\frac{dy}{dx} + 2y = x^2 - 1,$$

2. Solve the initial value problem:

$$\frac{d^2y}{dx^2} + 13\frac{dy}{dx} + 40y = 40x^2 + 146x + 241, \qquad y(0) = 10, \qquad y'(0) = -31.$$

3. Show that $y = x$ is a solution of the differential equation

$$(1+x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4y = 0.$$

Find another linearly independent solution to this equation.

4. Given that λ is a positive constant find the eigenvalues and eigenfunctions $\phi_n(x)$ for the boundary value problem:

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad y(0) = 0, \qquad y'(\pi) = 0.$$

Show that these eigenfunctions satisfy the orthogonality relation:

$$\int_0^{\pi} \phi_n(x)\phi_m(x)dx = 0 \qquad \text{for} \qquad n \neq m$$

5. Use a trial function of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 2xy = 0.$$

Write the general solution in the form $y = Af(x) + Bxg(x)$, where $A = y(0)$ and $B = y'(0)$. Write down the first 3 non zero terms of the expansions of $f(x)$ and $g(x)$.

Show that these solutions converge for all finite values of x.

6. Explain what is meant by the terms *ordinary point*, *singular point* and *regular singular point* for the differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0,$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials.

Find the singular points of the differential equation

$$x^3(x^2 - 3x + 2)\frac{d^2y}{dx^2} + x^2(x + 4)\frac{dy}{dx} + 5y = 0$$

and for each singular point state whether it is regular or not.

7. Find the general solution of the differential equations:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 6e^{2t} \\ 9e^{2t} \end{pmatrix}.$$

SECTION B

8. Show that when $\lambda \leq 4$ the boundary value problem

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + \lambda y = 0, \qquad y(0) = 0, \qquad y'(1) = 0. \tag{1}$$

has no eigenfunctions, but that for appropriate values of $\lambda > 4$, the eigenfunctions are:

$$\phi_n(x) = e^{-2x}\sin(\omega_n x), \qquad n = 1, 2, 3\cdots,$$

where ω_n satifies the equation $\omega_n = 2\tan\omega_n$.

Write eq(1) in Sturm Liouville form and hence or otherwise, show that

$$\int_0^\pi e^{4x}\phi_n(x)\phi_m(x)dx = 0 \qquad n \neq m.$$

9. Use a trial function of the form:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to find a series solution for the differential equation:

$$(1 - x^2)\frac{d^2y}{dx^2} + \lambda y = 0.$$

Show that the recurrence relation between the coefficients a_{n+2} and a_n is

$$\frac{a_{n+2}}{a_n} = \frac{n(n-1) - \lambda}{(n+1)(n+2)}.$$

Show that the general solution to this differential equation is a linear combination of a series of odd powers of x and a series of even powers of x

Show that if $\lambda = m(m-1)$ and m is an even positive integer, the even series solution terminates and is just a polynomial, while if m is an odd positive integer, the series of odd powers of x terminates and becomes a polynomial. Write down the polynomials for the cases when $m = 2, 3, 4, 5$. Denote these polynomials by $Q_m(x)$.

Show that for $m, n = 2, 3, 4, 5$,

$$\int_{-1}^{1} \frac{Q_n(x)Q_m(x)}{1 - x^2}dx = 0 \qquad \text{for all} \qquad m \neq n$$

10. Use a trial function of the form:

$$y = \sum_{n=0}^{\infty} a_n x^{n+c}$$

to find two linearly independent solutions of the differential equation:

$$4x\frac{d^2y}{dx^2} + (3-x)\frac{dy}{dx} - y = 0. \qquad (2)$$

Show that both of these solutions converge for all values of $x > 0$.

Write down the first three terms of each series.

Write eq(2) in Sturm Liouville form.

11. Show that the vector $(1, 2, -1)^T$ is an eigenvector for the matrix

$$A = \begin{pmatrix} 7 & 2 & 10 \\ -8 & -3 & -16 \\ 1 & 1 & 4 \end{pmatrix}.$$

Find the eigenvalue for this eigenvector. Find the other two eigenvalues and the corresponding eigenvectors.

Find a matrix P such that

$$P^{-1}AP = D,$$

where D is a diagonal matrix whose elements should be stated.

Transform the set of differential equations:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t)$$

into the form:

$$\frac{d\mathbf{y}}{dt} = D\mathbf{y} + \mathbf{c}(t),$$

where A is the matrix given above. Write down expressions for the components of $\mathbf{c}(t)$ in terms of the components of $\mathbf{f}(t)$.

12. Show that $\mathbf{x} = (1,3)^T e^{5t}$ is one solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 \\ -9 & 8 \end{pmatrix} \mathbf{x}.$$

Find a second solution and hence write down the general solution.

Find a linear transformation, $\mathbf{x} = P\,\mathbf{y}$, which will decouple the differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 \\ -9 & 8 \end{pmatrix} \mathbf{x} + \mathbf{f}(t),$$

where $\mathbf{f}(t)$ is some known function of t and write down the decoupled differential equations. Solve these differential equations and hence determine $\mathbf{x}(t)$ when $\mathbf{f}(t) = (0,1)^T e^{5t}$.