Solutions to MATH 201 JAN 2005

The solutions are all similar to questions set for homework except where marked bw for bookwork.

SECTION A

1. For the Complementary Function try a solution $y = e^{mx}$. This will satisfy the homogeneous equation if $m^2 + 13m + 36 = 0$. This has roots $m = -4$ and $m = -9$ so the Complementary Function is $Ae^{-4x} + Be^{-9x}$.

To find the particular integral we try $y = a\cos(6x) + b\sin(6x)$. We find

$$-36a\cos(6x) - 36b\sin(6x) + 13(-6a\sin(6x) + 6b\cos(6x)) + 36(a\cos(6x) + b\sin(6x)) = 156\cos(6x)$$

which has the solution $a = 0$, $b = 2$. The general solution is then

$$y = Ae^{-4x} + Be^{-9x} + 2\sin(6x)$$

From the initial conditions we get

$$y(0) = A + B = 5,$$

$$y'(0) = -4A - 9B + 12 = -13, \qquad \text{therefore} \qquad A = 4, \quad B = 1$$

The solution is therefore $y = 4e^{-4x} + e^{-9x} + 2\sin(6x)$

2. Substituting x in to the equation gives $2x - 2x = 0$

Try $y = xu$. We get

$$(1 + x^2)\left[xu'' + 2u'\right] - 2x\left[xu' + u\right] + 2xu = 0$$

This simplifies to

$$u'' = \left(\frac{-2}{x} + \frac{2x}{1+x^2}\right)u' \qquad \text{so that} \qquad \int \frac{d(u')}{u'} = -2\int \frac{dx}{x} + \int \frac{2x\,dx}{x^2+1}$$

$$\ln(u') = -2\ln x + \ln(x^2 + 1) \qquad \text{or} \qquad u' = \frac{1 + x^2}{x^2} = \frac{1}{x^2} + 1$$

Therefore

$$u = -\frac{1}{x} + x$$

The second solution is therefore $y = x^2 - 1$.

Sturm Liouville form is:

$$\frac{d}{dx}\left\{\frac{1}{x^2+1}\frac{dy}{dx}\right\} + \frac{2}{(x^2+1)^2}y = 0$$

3. The general solution to the differential equation is $y = A\cos(\omega x) + B\sin(\omega x)$. $y(0) = 0 = A$ and $y(5) = 5 = B\sin(5\omega)$. Thus $B = 5/\sin(5\omega)$ provided that $\sin(5\omega) \neq 0$. If $\sin 5\omega = 0$ there is no solution. This will occur if $5\omega = n\pi$ for any integer n. That is there is no solution for $\lambda = n^2\pi^2/25$.

4. For $\lambda = \omega^2$, $y = A\cos(\omega x) + B\sin(\omega x)$. $y(0) = A = 0$ and $y'(2) = B\omega\cos(2\omega)$. $B = 0$ unless $\cos(2\omega) = 0$ that is if $\omega = (n + 1/2)\pi/2$. The eigenvalues are therefore $\lambda = (n + 1/2)^2\pi^2/4$ and the eigenfunctions $\sin((n + 1/2)\pi x/2)$.

5.

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + ... + (n+2)(n+1)a_{n+2}x^n + ...$$

$$x\frac{dy}{dx} = \qquad a_1x + 2a_2x^2 + 3a_3x^3 + ...na_nx^n + ...$$

$$3y = \quad 3a_0 + 3a_1x + 3a_2x^2 + 3a_3x^3 + ... + 3a_nx^n + ...$$

We get a recurrence relation between a_{n+2} and a_n by equating coefficients of x^n. We find that

$$\frac{a_{n+2}}{a_n} = \frac{n+3}{(n+2)(n+1)}.$$

Thus all of the even coefficients are determined in terms of a_0 and all of the odd coefficients in terms of a_1.

$$f(x) = 1 + \frac{3}{2}x^2 + \frac{5}{3.4}\frac{3}{2}x^4 + ...$$

and

$$g(x) = 1 + \frac{4}{3.2}x^2 + \frac{4}{3.2}\frac{6}{5.4}x^4 + ...$$

The ratio between successive terms in the two series is $(n+3)x^2/((n+2)(n+1))$. The size of this ratio tends to zero for all finite values of x as n tends to ∞. The two series will therefore converge for all finite values of x.

6. Any point where $P(x) = 0$ and so d^2y/dx^2 is not defined is a singular point of the differential equation. All other points are ordinary points. If x_0 is a singular point and

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \quad \text{exists} \quad \text{and} \quad \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} \quad \text{exists}$$

the singular point is a regular singular point. If either limit does not exist, the point $x = x_0$ is an irregular singular point.

There are 3 singular points, $x = 0$, $x = -4$ and $x = 4$.

For $x = 0$ we have $(x+4)/(x^2-16)^2$ is finite, but $(x^2+1)/[x(x^2-16)^2]$ is not as $x \to 0$, so $x = 0$ is an irregular singular point.

For $x = -4$, $x^2(x+4)^2/[x^3(x^2-16)^2]$ and $(x+4)^2(x^2+1)/[x^3(x^2-16)^2]$ are both finite as $x \to -4$, so this point is a regular singular point

For $x = 4$, we have $(x-4)Q(x)/P(x) = x^2(x-4)(x+4)/[x^3(x+4)^2(x-4)^2]$. This does not tend to a finite limit as $x \to 4$ and so this point is an irregular singular point.

7. The eigenvalues of A satisfy $(3-\lambda)(1-\lambda)-24=0$ so the eigenvalues are $\lambda=-3$ and $\lambda=7$. The eigenvectors are $(1,-1)^T$ and $(3,2)^T$. The complementary function is therefore $A_1(1,-1)^T e^{-3t}+A_2(3,2)^T e^{7t}$, where A_1 and A_2 are constants.

The particular integral will be $\mathbf{w}e^{2t}$. Substituting into the differential equation and dividing through by e^{2t} gives:

$$2w_1 = 3w_1 + 6w_2 - 20 \qquad \text{and} \qquad 2w_2 = 4w_1 + w_2 - 5.$$

The solution is $w_1 = 2$ and $w_2 = 3$.

The complete solution is then

$$\mathbf{x} = A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + A_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{7t} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t},$$

where A_1 and A_2 are constants

SECTION B

8. For $\lambda=-p^2$, the general solution is $y = Ae^{(p-3)x}+Be^{-(p+3)x}$. The boundary conditions give $y(0)=0=A+B$ and $y(\pi)=0=Ae^{\pi(p-3)}+Be^{-\pi(p+3)}$. The only solution is $A=B=0$.

For $\lambda=0$ the general solution is $y=(A+Bx)e^{-3x}$. The boundary conditions give $y(0)=0=A$ and $y(\pi)=0=\pi Be^{-3\pi}$ and so $A=B=0$.

For $\lambda=\omega^2$, we have $y=e^{-3x}\Big(A\cos(\omega x)+B\sin(\omega x)\Big)$.

The boundary conditions give $y(0)=0=A$ and then $y(\pi)=0=Be^{-3\pi}\sin(\pi\omega)$. Thus $B=0$ unless $\sin(\pi\omega)=0$. This will be so if $\omega=n$. The eigenvalues are therefore $\lambda=n^2$ and the eigenfunctions $e^{-3x}\sin(nx)$.

$$\int_0^\pi e^{6x}\phi_n(x)\phi_m(x)dx = \int_0^\pi \sin(nx)\sin(mx)dx$$

$$= \frac{1}{2}\int_0^\pi \Big\{\cos\Big((n-m)x\Big)-\cos\Big((n+m)x\Big)\Big\}dx = \frac{1}{2}\left[\frac{\sin((n-m)x}{(n-m)} - \frac{\sin((n+m)x}{(n+m)}\right]_0^\pi = 0$$

$$\int_0^\pi \sin^2(nx)dx = \frac{1}{2}\int_0^\pi \Big\{1-\cos(2nx)\Big\}dx = \frac{\pi}{2}.$$

9. The coefficient of x^n in the differential equation is

$$(n+2)(n+1)a_{n+2} - 2na_n + \lambda a_n = 0$$

This gives the recurrence relation

$$\frac{a_{n+2}}{a_n} = \frac{2n-\lambda}{(n+2)(n+1)}.$$

Clearly if $\lambda = 2m$ for some integer m, a_{m+2}/a_m=0, and this series terminates. If m is odd, it is the series of odd powers which terminates, while if m is even, the series of even powers terminates.

If $m = 1$, $a_3 = 0$, so $H_1(x) = x$.
If $m = 2$, $a_4 = 0$, so $H_2(x) = 1 - 2x^2$.
If $m = 3$, $a_5 = 0$, so $H_3(x) = x - 2x^3/3$.

$$\frac{d}{dx}\left\{e^{-x^2}\frac{dy}{dx}\right\} + e^{-x^2}\lambda y = e^{-x^2}\frac{d^2y}{dx^2} - 2xe^{-x^2}\frac{dy}{dx} + e^{-x^2}\lambda y.$$

Divide this by $\exp -x^2$ and we get eq(1).

$H_1(x)$ and $H_3(x)$ are odd functions of x and $H_2(x)$ is an even function. This means that the integrals with $H_1(x)H_2(x)$ and $H_3(x)H_2(x)$ are zero. The only integral we have to evaluate is then

$$\int_{-\infty}^{\infty} e^{-x^2} x(x - 2x^3/3)dx. \tag{2}$$

Consider

$$\int_{-\infty}^{\infty} e^{-x^2} x2x^3/3dx = -\int_{-\infty}^{\infty} x^3/3d\big(e^{-x^2}\big) = \Big[e^{-x^2}x^3/3\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2}x^2dx.$$

Thus the integral (2) is zero

10. Substituting into the differential equation:

$$7c(c-1)a_0x^{c-1} + 7(c+1)ca_1x^c + ... + 7(c+n+1)(c+n)a_{n+1}x^{n+c} + ...$$

$$+6ca_0x^{c-1} + 6(c+1)a_1x^c + ... + 6(c+n+1)a_{n+1}x^{n+c} + ...$$

$$-ca_0x^c - (c+1)a_1x^{c+1} - ... - (c+n)a_nx^{c+n} - ...$$

$$-a_0x^c - a_1x^{c+1} - ... - a_nx^{c+n}.$$

The smallest power of x in this is x^{c-1}. Its coefficient is $(7c(c-1) + 6c)a_0$. a_0 is not zero by hypothesis and so $c = 0$ or $c = 1/7$.

We get the recurrence relation by looking at the coefficient of x^{n+c}. This gives us

$$\frac{a_{n+1}}{a_n} = \frac{c+n+1}{(c+n+1)(7c+7n+6)} = \frac{1}{7c+7n+6}$$

For $c = 0$ if we take $a_0 = 1$,

$$y = 1 + \frac{1}{6}x + \frac{1}{6}\frac{1}{13}x^2$$

and for $c = 1/7$ we have

$$y = x^{\frac{1}{7}}\left\{1 + \frac{1}{7}x + \frac{1}{7}\frac{1}{14}x^2 + ...\right\}$$

11. $A(1,0,-1)^T = (2,0,-2)^T = 2(1,0,-1)^T$, so $(1,0,-1)^T$ is an eigenvector with eigenvalue $\lambda = 2$.

$A(2,1,0)^T = (8,4,0)^T = 4(2,1,0)^T$, so $(2,1,0)^T$ is an eigenvector with eigenvalue $\lambda = 4$.

The third eigenvector can be found by multiplying the first column of $A - \lambda_2 I$ by the matrix $A - \lambda_1 I$. That is

$$\mathbf{e}_3 = \begin{pmatrix} -5 & 14 & -5 \\ -3 & 8 & -3 \\ -5 & 10 & -5 \end{pmatrix} \begin{pmatrix} -7 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 18 \\ 12 \\ 30 \end{pmatrix}.$$

We can take a factor of 6 out so that $\mathbf{e}_3 = (3,2,5)^T$. Its eigenvalue is -2.

Alternatively, we can find the other eigenvalue from the determinant

$$\det \begin{matrix} -3-\lambda & 14 & -5 \\ -3 & 10-\lambda & -3 \\ -5 & 10 & -3-\lambda \end{matrix} = 0 = -\lambda^3 + 4\lambda^2 + 4\lambda - 16 = -(\lambda - 4)(\lambda^2 - 4).$$

The roots are $\lambda = 4$ and $\lambda = \pm 2$. The other eigenvector can be found by taking the cofactors of the elements of the first row of $A + 2I$. This gives a multiple of $(3,2,5)^T$.

The matrix P is the matrix whose columns are the eigenvectors, so

$$P = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 5 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Write $\mathbf{x} = P\mathbf{y}$ and then $\dot{\mathbf{x}} = P\dot{\mathbf{y}} = AP\mathbf{y} + \mathbf{f}(t)$.

Then $P^{-1}P\dot{\mathbf{y}} = P^{-1}AP\mathbf{y} + P^{-1}\mathbf{f}(t)$
or $\dot{\mathbf{y}} = D\mathbf{y} + \mathbf{c}(t)$, where $\mathbf{c}(t) = P^{-1}\mathbf{f}(t)$. The inverse matrix P^{-1} is

$$\begin{pmatrix} 5/4 & -5/2 & 1/4 \\ -1/2 & 2 & -1/2 \\ 1/4 & -1/2 & 1/4 \end{pmatrix}$$

Therefore $c_1 = 5f_1/4 - 5f_2/2 + f_3/4$, $c_2 = -f_1/2 + 2f_2 - f_3/2$ and $c_3 = f_1/4 - f_2/2 + f_3/4$.

12.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^{3t} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

We look for a second solution $(1,1)^T te^{3t} + \mathbf{w}e^{3t}$. We get

$$\begin{pmatrix} 3t+1+3w_1 \\ 3t+1+3w_2 \end{pmatrix} e^{3t} = \begin{pmatrix} 3t+2w_1+w_2 \\ 3t-w_1+4w_2 \end{pmatrix} e^{3t}.$$

Therefore

$$w_1 - w_2 = -1 \qquad \text{and} \qquad w_1 - w_2 = -1.$$

We can take $w_2 = 1$ and $w_1 = 0$. The second solution is then $y = (1,1)^T te^{3t} + (0,1)^T e^{3t}$
The general solution is then $y = A(1,1)^T e^{3t} + B[(1,1)^T te^{3t} + (0,1)^T e^{3t}]$

We take P to have the first column $(1,1)^T$ and second column $(0,1)^T$ Thus

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Writing $\mathbf{x} = P\mathbf{y}$

$$\frac{d\mathbf{y}}{dt} = P^{-1}\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbf{f} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbf{f}$$

The equations are now decoupled. The equation for y_2 does not involve y_1.

We have $\dot{y}_1 = 3y_1 + y_2 + f_1$ and $\dot{y}_2 = 3y_2 - f_1 + f_2$

We have to solve $\dot{y}_2 = 3y_2 + e^{3t}$. This is a first order linear equation whose integrating factor is e^{-3t}. The solution is $y_2 = (A+t)e^{3t}$.

The equation for y_1 is $\dot{y}_1 = 3y_1 + (A+t)e^{3t}$. This is also a first order linear equation with integrating factor e^{-3t}. The solution is $y_1 = (B + At + t^2/2)e^{3t}$. Then

$x_1 = y_1 = (B + At + t^2/2)e^{3t}$, and $x_2 = -y_1 + y_2 = (A - B - (A-1)t - t^2/2)e^{3t}$.