MATH 201 Jan 2005

ORDINARY DIFFERENTIAL EQUATIONS

TIME ALLOWED: TWO HOURS AND A HALF

Instructions to candidates

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

SECTION A

1. Solve the initial value problem:

$$\frac{d^2y}{dx^2} + 13\frac{dy}{dx} + 36y = 156\cos(6x), \qquad y(0) = 5, \qquad y'(0) = -13.$$

2. Show that $y = x$ is a solution of the differential equation

$$(1 + x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

Find another linearly independent solution to this equation.

Write the differential equation above in the Sturm-Liouville self adjoint form.

3. Solve the two point boundary value problem:

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad y(0) = 0, \qquad y(5) = 5,$$

given that $\lambda = \omega^2$ is a positive constant. Show that it is not possible to find a solution for all values of λ and find the values of λ for which there is no solution.

4. Given that λ is a positive constant find the eigenvalues and eigenfunctions for the boundary value problem:

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad y(0) = 0, \qquad y'(2) = 0.$$

5. Use a trial function of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to find the solution of the differential equation

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} + 3y.$$

Write the general solution in the form $y = Af(x) + Bxg(x)$, where $A = y(0)$ and $B = y'(0)$. Write down the first 3 non zero terms of the expansions of $f(x)$ and $g(x)$.

Show that these solutions converge for all finite values of x.

6. Explain what is meant by the terms *ordinary point*, *singular point* and *regular singular point* for the differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0,$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials.

Find the singular points of the differential equation

$$x^3(x^2 - 16)^2\frac{d^2y}{dx^2} + x^2(x+4)\frac{dy}{dx} + (x^2+1)y = 0$$

and for each singular point state whether it is regular or not.

7. Find the general solution of the differential equations:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & 6 \\ 4 & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 20e^{2t} \\ 5e^{2t} \end{pmatrix}.$$

SECTION B

8. Show that when $\lambda \leq 0$ the boundary value problem

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + (9+\lambda)y = 0, \qquad y(0) = 0, \qquad y(\pi) = 0$$

has no eigenfunctions, but that for appropriate values of $\lambda > 0$, the eigenfunctions are:

$$\phi_n(x) = e^{-3x}\sin(nx), \qquad n = 1, 2, 3\cdots.$$

Further, show that

$$\int_0^\pi e^{6x}\phi_n(x)\phi_m(x)dx = 0 \qquad n \neq m$$

and evaluate

$$\int_0^\pi e^{6x}\phi_n^2(x)dx.$$

9. Use a trial function of the form:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to find a series solution for the differential equation:

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0. \tag{1}$$

Show that the recurrence relation between the coefficients a_{n+2} and a_n is

$$\frac{a_{n+2}}{a_n} = \frac{2n-\lambda}{(n+1)(n+2)}.$$

Show that if $\lambda = 2m$, where m is a positive integer, one of these series terminates and is a finite polynomial. Write down this polynomial for each of the cases when $m = 1, 2, 3$. Denote these polynomials by $H_m(x)$.

Show that the differential equation

$$\frac{d}{dx}\left(e^{-x^2}\frac{dy}{dx}\right) + e^{-x^2}\lambda y = 0$$

is the Sturm-Liouville self adjoint form of equation (1).

Show that for $n, m = 1, 2, 3$, the integrals

$$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)\,dx = 0 \qquad n \neq m.$$

10. Use a trial function of the form:

$$y = \sum_{n=0}^{\infty} a_n x^{n+c}$$

to find two linearly independent solutions of the differential equation:

$$7x\frac{d^2y}{dx^2} + (6-x)\frac{dy}{dx} - y = 0.$$

Write down the first three terms of each series.

11. Show that the vectors $(1,0,-1)^T$ and $(2,1,0)^T$ are eigenvectors for the matrix

$$A = \begin{pmatrix} -3 & 14 & -5 \\ -3 & 10 & -3 \\ -5 & 10 & -3 \end{pmatrix}.$$

Find the eigenvalues for these two eigenvectors. Find the other eigenvector and the corresponding eigenvalue.

Find a matrix P such that

$$P^{-1}AP = D,$$

where D is a diagonal matrix whose elements should be stated.

Transform the set of differential equations:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t)$$

into the form:

$$\frac{d\mathbf{y}}{dt} = D\mathbf{y} + \mathbf{c}(t),$$

where A is the matrix given above. Write down expressions for the components of $\mathbf{c}(t)$ in terms of the components of $\mathbf{f}(t)$.

12. Show that $\mathbf{x} = (1, 1)^T e^{3t}$ is one solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x}.$$

Find a second solution and hence write down the general solution.

Find a linear transformation, $\mathbf{x} = P\,\mathbf{y}$, which will decouple the differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x} + \mathbf{f}(t),$$

where $\mathbf{f}(t)$ is some known function of t and write down the decoupled differential equations. Solve these differential equations and hence determine $\mathbf{x}(t)$ when $\mathbf{f}(t) = (0, 1)^T e^{3t}$.