PAPER CODE NO.
MATH199

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THE UNIVERSITY
of LIVERPOOL

MAY 2006 EXAMINATIONS

Bachelor of Engineering : Year 1
Bachelor of Science : Year 2
Master of Engineering : Year 1

MATHEMATICAL TECHNIQUES FOR ENGINEERS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to Section A and to the best THREE questions from Section B will be taken into account. Section A carries 55% of the available marks.

Your attention is drawn to the formula sheet which accompanies this exam paper.

SECTION A

1. Differentiate the following functions with respect to x, simplifying your answers as much as possible:

$$\text{(i)}\; x^3e^{2x}, \qquad \text{(ii)}\; 3x^2\sin(4x) + x\cos(4x), \qquad \text{(iii)}\; \frac{2x-1}{(x+2)^2}.$$

[6 marks]

2. Sketch the graph of $y = x(x^2 - 1)$. Mark the coordinates of any points where the graph crosses the axes and any stationary points. [6 marks]

3. Evaluate:

$$\text{(i)}\; \int (x^3 - 2x^2 + 5)dx\,, \qquad \text{(ii)}\; \int \frac{7}{\sqrt{x^2-25}}dx\,, \qquad \text{(iii)}\; \int_0^\infty 4e^{-2x}dx\,.$$

[6 marks]

4. Use the substitution $u = 3x^2 - 4$ to evaluate the indefinite integral

$$\int 2x\cos(3x^2-4)dx\,.$$

[3 marks]

5. Solve the differential equation

$$\frac{dy}{dt} = 7y$$

given that $y = 3$ when $t = 0$. [3 marks]

6. Find the value of the number λ such that the vector $2\mathbf{i} + \lambda\mathbf{j} - \mathbf{k}$ is orthogonal to $\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. [3 marks]

7. Given the vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$,

   (i) find the vector $3\mathbf{a} - 2\mathbf{b}$ and determine its magnitude to 2 decimal places;

   (ii) find the angle, to the nearest degree, between $3\mathbf{a} - 2\mathbf{b}$ and $\mathbf{a}$;

   (iii) evaluate $|\mathbf{a} \times \mathbf{b}|$ to 2 decimal places.

[7 marks]

8. The vertices O, A, B of a triangle have coordinates $(0, 0, 0)$, $(1, -2, 1)$ and $(2, 0, -3)$ respectively. Calculate the area of triangle OAB to 2 decimal places. [5 marks]

9. Given the complex numbers $z_1 = 3 + 2i$ and $z_2 = -1 + 5i$, find

$$z_1 + z_2\,, \qquad z_1 z_2\,, \qquad \text{and} \qquad \frac{z_1}{z_2}\,,$$

expressing your answer in the form $a + ib$, where a and b are real numbers. [6 marks]

10. Sketch the level curves $w = 0, 1$ and 2 of the function $w = y - 4x^2 + 1$. [4 marks]

11. Find all first order and second order partial derivatives with respect to x and y of the function

$$w = xe^{-y} + x^2 y\,,$$

and verify that

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}\,.$$

[6 marks]

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SECTION B

12. Given that

$$y = \frac{2x^2}{x^2 - 9}$$

show that its derivative can be written as

$$y' = -\frac{36x}{(x^2 - 9)^2}\,.$$

[3 marks]

Sketch the graph of y. Include on your sketch the coordinates of any points where the curve crosses the axes, the coordinates of any stationary points and the equations of any asymptotes. [12 marks]

13. (i) Solve the differential equation

$$\frac{dy}{dx} = 7y + \sin(3x)\,.$$

[4 marks]

(ii) The four edges of a uniform metal plate are defined by the curve $y = 4 - 3x^2$, the x-axis, the y-axis and the line $x = 1$. Sketch the plate and determine its area. Find the coordinates $(\bar{x}, \bar{y})$ of its centroid. [11 marks]

Reminder: expressions for $(\bar{x}, \bar{y})$ *are given in the formula sheet.*

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14. The coordinates of the points A, B and C are $(-1, 0, 3)$, $(-5, 4, 4)$ and $(3, 2, -4)$ respectively.

    (i) Write down the line vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. [3 marks]

    (ii) Hence find, to the nearest degree, the angle between the lines AB and BC. [6 marks]

    (iii) Find the vector form of the equation of the straight line which passes through the points B and C. [3 marks]

    (iv) Determine whether the point $(-1, 3, 0)$ lie on this line, justifying your answer. [3 marks]

15. (i) Sketch the region $|z - 1 + 2i| = 3$ in the complex plane where $z = x + iy$. [3 marks]

    (ii) Working to three decimal places, express $-1 + 2i$ and $2 - 3i$ in a polar form. Hence find a polar form for

$$\frac{(2-3i)^3}{(-1+2i)^2}.$$

[6 marks]

    (iii) Given the harmonic function

$$V(t) = 5\cos(\frac{\pi t}{6}) + 12\sin(\frac{\pi t}{6})$$

write down its amplitude and frequency. Express $V(t)$ as a cosine harmonic function, evaluating its phase angle to the nearest degree. [6 marks]

16. Sketch the level curves $w = -1$, $w = 0$ and $w = 1$ of the function $w = -2y + 6x^2$. [5 marks]

    (i) Find the rate of change of w at the point $(2, 1)$, in the outward radial direction. [5 marks]

    (ii) Find the rate of change of w at the point $(3, 2)$, in the direction towards $(4, 5)$. [5 marks]

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MATH199 Formulæ Sheet

Differentiation Rules.
Function of a function:

$$\frac{d}{dx} f\left[g(x)\right] \;=\; \frac{dg}{dx}\frac{df}{dg}\ .$$

Product Rule:

$$\frac{d}{dx}(uv) \;=\; \frac{du}{dx}v \;+\; u\frac{dv}{dx}\ .$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) \;=\; \left(\frac{du}{dx}v \;-\; u\frac{dv}{dx}\right)\frac{1}{v^2}\ .$$

Function	Derivative	Function	Derivative
cx^n	ncx^{n-1}	$cg^n(x)$	$cng^{n-1}(x)\ \frac{dg}{dx}$
$\frac{c}{x}$	$-\ \frac{c}{x^2}$	$\frac{c}{g(x)}$	$-\ \frac{c}{g^2(x)}\frac{dg}{dx}$
$\frac{c}{x^n}$	$-\ \frac{cn}{x^{n+1}}$	$\frac{c}{g^n(x)}$	$-\ \frac{cn}{g^{n+1}(x)}\frac{dg}{dx}$
ce^x	ce^x	$ce^{g(x)}$	$ce^{g(x)}\ \frac{dg}{dx}$
$c\ln x$	$\frac{c}{x}$	$c\ln g(x)$	$\frac{c}{g(x)}\frac{dg}{dx}$
$\cos x$	$-\ \sin x$	$\cos g(x)$	$-\ \frac{dg}{dx}\ \sin g(x)$
$\sin x$	$\cos x$	$\sin g(x)$	$\frac{dg}{dx}\ \cos g(x)$

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To Sketch the Graph of $y = f(x)$.

1. Find the value of y when $x = 0$.
2. Find the value(s) of x for $y = 0$ by solving $f(x) = 0$, if solutions exist.
3. Find and classify stationary points, if any.
4. For rational functions $y(x) = \dfrac{u(x)}{v(x)}$, also find (a) vertical asymptotes, if any, by solving $v(x) = 0$, (b) behaviour at infinity by dominant term method and long division.
5. If $f(x)$ involves exponentials find behaviour at infinity by using the mnemonic $e^{+\infty} = +\infty$ and $e^{-\infty} = 0$ and that exponentials dominate any power of x.

List of Integrals.

Integrand	Integral
$(ax+b)^n$	$\dfrac{(ax+b)^{n+1}}{a(n+1)}$ for $n \neq -1$
$\dfrac{1}{(ax+b)}$	$\dfrac{1}{a}\ln\lvert ax+b\rvert$
$\dfrac{1}{(ax+b)(cx+d)}$	$\dfrac{1}{(ad-bc)}\ln\left\lvert\dfrac{(ax+b)}{(cx+d)}\right\rvert$ for $ad - bc \neq 0$
$\dfrac{1}{(ax^2-b)}$ for $a, b > 0$	$\dfrac{1}{2\sqrt{ab}}\ln\left\lvert\dfrac{(\sqrt{a}x-\sqrt{b})}{(\sqrt{a}x+\sqrt{b})}\right\rvert$
$\dfrac{1}{(ax^2+b)}$ for $a, b > 0$	$\dfrac{1}{\sqrt{ab}}\tan^{-1}\left(\sqrt{\dfrac{a}{b}}x\right)$
$\dfrac{1}{\sqrt{(a^2x^2+b^2)}}$	$\dfrac{1}{a}\sinh^{-1}\left(\dfrac{ax}{b}\right)$
$\dfrac{1}{\sqrt{(a^2x^2-b^2)}}$	$\dfrac{1}{a}\cosh^{-1}\left(\dfrac{ax}{b}\right)$
$\dfrac{1}{\sqrt{(b^2-a^2x^2)}}$	$\dfrac{1}{a}\sin^{-1}\left(\dfrac{ax}{b}\right)$
$e^{ax}\cos bx$	$\dfrac{1}{(a^2+b^2)}e^{ax}(a\cos bx + b\sin bx)$
$e^{ax}\sin bx$	$\dfrac{1}{(a^2+b^2)}e^{ax}(a\sin bx - b\cos bx)$

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Integration by parts.

$$\int u(x)v(x)\,dx \;=\; u(x)A \;-\; \int \frac{du}{dx}A\,dx \qquad \text{where} \quad A \;=\; \int v(x)dx\ .$$

Applications of the Definite Integral.

The area between the curve $y = f(x)$ and the x axis from $x = a$ to $x = b$, where $b > a$, is

$$\int_a^b |f(x)|\,dx\ .$$

The coordinates $(\bar{x}, \bar{y})$ of the centroid of a uniform sheet bounded by curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ are given by

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx\,, \qquad \bar{y} = \frac{1}{2A}\int_a^b [(f(x))^2 - (g(x))^2]dx\,.$$

where the area A is given by

$$A = \int_a^b [f(x) - g(x)]dx\,.$$

The mean (average) value of the function $f(x)$ over the interval $a \le x \le b$ is

$$\frac{1}{(b-a)}\int_a^b f(x)\,dx\ .$$

The area swept out by the polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$\frac{1}{2}\int_\alpha^\beta r^2\,d\theta\ .$$

Differential Equations.

Equation	Solution
$\dfrac{dy}{dx} = f(x)$	$y = \int f(x)\,dx$
$\dfrac{dy}{dx} = ay$	$y = Ce^{ax}$
$\dfrac{dy}{dx} = ay + f(x)$	$y = e^{ax}\left[C + \int e^{-ax}f(x)\,dx\right]$
$\dfrac{d^2y}{dx^2} = ay \quad \text{if } a > 0$	$y = Ae^{\sqrt{a}\,x} + Be^{-\sqrt{a}\,x}$
$\dfrac{d^2y}{dx^2} = ay \quad \text{if } a < 0$	$y = A\cos\left(\sqrt{\lvert a\rvert}\,x\right) + B\sin\left(\sqrt{\lvert a\rvert}\,x\right)$

Vector Algebra.
Given $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$

$$|\mathbf{a}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}\ ,\ \cos\alpha = \frac{a_1}{|\mathbf{a}|}\ ,\ \cos\beta = \frac{a_2}{|\mathbf{a}|}\ ,\ \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

where α, β and γ are the angles between the vector **a** and the positive x, y and z directions respectively.
$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$ where θ is the angle between **a** and **b** and where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.
a and **b** are orthogonal if $\mathbf{a}\cdot\mathbf{b} = 0$.

$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$
$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\,\sin\theta$, where $0 \le \theta < \pi$.

The area of a triangle with sides given by vectors **a** and **b** has magnitude

$$\frac{1}{2}|\mathbf{a}\times\mathbf{b}|\,.$$

Vector Geometry of Lines.
The vector line $\overrightarrow{AB}$ is the vector whose magnitude is the length of the line AB and whose direction is from point A to point B

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}\,.$$

The vector (parametric) equation of the straight line passing through points A and B is $\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$ where λ is a parameter.

Forces on a body in equilibrium.
For a body to remain in equilibrium while acted on by forces $\mathbf{F}_1$, $\mathbf{F}_2$,..., $\mathbf{F}_n$ through points with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$,...,$\mathbf{r}_n$, the forces $\mathbf{F}_i$ and moments $\mathbf{M}_i = \mathbf{r}_i\times\mathbf{F}_i$ must satisfy

$$\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = \mathbf{0}\,,\qquad \mathbf{M}_1 + \mathbf{M}_2 + \ldots + \mathbf{M}_n = \mathbf{0}\,.$$

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Complex Numbers.
Polar form: $x + iy = r\cos\theta + ir\sin\theta$, where r and θ are the polar coordinates of (x, y).
Exponential form: $x + iy = re^{i\theta}$ where θ must be in radians.
Power theorem: $(x + iy)^n = r^n\cos(n\theta) + ir^n\sin(n\theta)$.
Modulus: $|(x + iy)| = r = \sqrt{(x^2 + y^2)}$
Argument: $\arg(x + iy) = \theta$ where $-\pi < \theta \leq \pi$.

Complex Functions.
If $w = f(z)$ where $w = u + iv$ and $z = x + iy$ then u and v are functions of x and y which satisfy the two dimensional Laplace equation for all values of x and y at which w and all its derivatives exist.

Complex Mapping.
$\arg z = \alpha$ is the half line from the origin and inclined at an angle α to the positive x axis.
$|z - z_1| = c$ is the circle with centre at point z_1 and radius c.

Harmonic Functions.
Given the harmonic functions $V = a\cos(\omega t) + b\sin(\omega t)$, the amplitude is $A = \sqrt{(a^2 + b^2)}$, the period is $2\pi/\omega$ and the frequency is $\omega/(2\pi)$.
V is equal to the cosine harmonic function $A\cos(\omega t - \epsilon)$ where $\epsilon = \arg(a + ib)$.
The harmonic function is also equal to the real part of the complex harmonic function $Ae^{i(\omega t - \epsilon)}$.
V satisfies

$$\frac{d^2V}{dt^2} = -\omega^2 V .$$

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Functions of Two Variables $w = f(x,y)$.
The level curves of w are the curves $f(x,y) = c$ where c is a parameter.
Taylor's theorem

$$\begin{aligned} f(x+h, y+k) &= f(x,y) + \left\{ \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k \right\} \\ &\quad + \frac{1}{2} \left\{ \frac{\partial^2 f}{\partial x^2} h^2 + 2 \frac{\partial^2 f}{\partial x \partial y} hk + \frac{\partial^2 f}{\partial y^2} k^2 \right\} + \ldots \end{aligned}$$

Application 1: The rate of change of $w = f(x,y)$ at a point and in the direction inclined at an angle θ to the positive x axis is the value of

$$\frac{\partial w}{\partial x} \cos\theta + \frac{\partial w}{\partial y} \sin\theta$$

at that point.
Application 2: At a stationary point $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are both simultaneously equal to zero. If

$$\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2$$

< 0 then the point is a saddle point;

> 0 and $\frac{\partial^2 w}{\partial x^2} < 0$ then it is a local maximum;

> 0 and $\frac{\partial^2 w}{\partial x^2} > 0$ then it is a local minimum;

$= 0$ the test is inconclusive.

Application 3: Small errors/fluctuations

$$\Delta w = \left| \frac{\partial w}{\partial x} \Delta x \right| + \left| \frac{\partial w}{\partial y} \Delta y \right| .$$