PAPER CODE NO.
MATH195

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THE UNIVERSITY
of LIVERPOOL

JANUARY 2005 EXAMINATIONS

Bachelor of Engineering : Year 1
Master of Engineering : Year 1

MATHEMATICS I FOR CIVIL ENGINEERS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 52% of the available marks.

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SECTION A

1. Express the following expression as a single logarithm

$$\tfrac{1}{2}\ln 36 - 2\ln 15 + \ln 2 + \ln 75\,.$$

[**2 marks**]

2. Differentiate the following functions with respect to x, noting any rules of differentiation which you use:

$$\text{(i) } x\cos 3x\,, \qquad \text{(ii) } \tan^2 x\,, \qquad \text{(iii) } \frac{(x-2)}{(x^2+1)}\,, \qquad \text{(iv) } \cosh(x^5)\,.$$

[**6 marks**]

3. Express the function

$$f(x) = \frac{2x^2 + x - 3}{x+3}\,, \qquad x \neq -3$$

in the form

$$f(x) = Ax + B + C/(x+3)$$

and hence indentify any asymptotes which this function may have. [*Note: you are not required to graph the function or identify any stationary values.*]

Would the function cross the x-axis if plotted and if so where?

[**7 marks**]

4. The function f is defined by

$$f(x) = \frac{5x+4}{x-3}\,, \qquad \text{for} \quad x \neq 3\,.$$

Find the inverse function $f^{-1}(x)$ and verify that $f^{-1}[f(x)] = x$.

[**4 marks**]

5. State L'Hôpital's rule for the evaluation of limits. Hence, or otherwise, evaluate
$$\lim_{x\to 0} \frac{1 + xe^{-x} - \cos 2x}{3x^2 - x}\,.$$
[5 marks]

6. Evaluate the following limits, where they exist, or show that they do not exist:
$$\text{(i)}\ \lim_{x\to 1} \frac{3x^2 - x - 2}{4x^2 + 2x - 6}\,, \qquad \text{(ii)}\ \lim_{x\to 2} \frac{3x^2 + x - 3}{x - 2}\,, \qquad \text{(iii)}\ \lim_{x\to\infty} \frac{4x + 6}{x^2}\,.$$
[6 marks]

7. Given that $x^2 e^y + x^3 y^2 = \sin 2y$, find $\frac{dy}{dx}$ as a function of x and y. **[4 marks]**

8. Evaluate the following indefinite integrals:
$$\text{(i)}\ \int xe^x dx\,, \qquad \text{(ii)}\ \int x^2 \ln x dx\,, \qquad \text{(iii)}\ \int \frac{t-2}{t+3} dt\,.$$
[7 marks]

9. The vectors **a** and **b** are the position vectors (relative to the origin O) of the points A and B respectively, with coordinates $A\,(1,-3,4)$ and $B\,(1,-2,3)$. Calculate
$$\text{(i)}\,\mathbf{a}\cdot\mathbf{b}\,, \qquad \text{(ii)}\,\mathbf{a}\times\mathbf{b}\,.$$
[5 marks]

10. Write down the *vector* equation of the straight line through the points $(-1,3,2)$ and $(2,0,6)$.

    Find the coordinates of the point where this line passes through the xy plane (i.e. where $z = 0$).

    [6 marks]

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SECTION B

11. Find and classify all stationary points of the function f defined by

$$f(x) = \frac{x^2 - 9x + 18}{x - 2}, \qquad x \neq 2\,.$$

(Note that for full credit, all mathematical working must be shown.)

Sketch the graph of $y = f(x)$, showing clearly any turning points, asymptotes and any points at which the graph intersects the x and y axes.

Find where the curve intersects the line $y = 3 - x$.

[**16 marks**]

12. (i) Given the complex numbers $z_1 = 3+2i$ and $z_2 = -1+5i$, compute

$$z_1 + z_2\,, \qquad z_1 z_2\,, \qquad \text{and} \qquad \frac{z_1}{z_2}\,,$$

expressing your answer in the form $a + ib$, where a and b are real numbers.

(ii) Write the complex number $-1 + i$ in modulus-argument form. Hence find all complex numbers z which satisfy $z^3 + 1 = i$. Sketch a diagram showing the position of these complex numbers in the complex plane.

(iii) Express

$$i^{25}(-1 + i)^{12}$$

in the form $a + ib$, with a and b real numbers.

[**16 marks**]

13. Write down the definitions of $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}.

Use these definitions to show that

$$2\cosh^2 x = 1 + \cosh 2x \qquad \text{and} \qquad 2\sinh x\cosh x = \sinh 2x\,.$$

A heavy cable of uniform density hangs under gravity taking the shape of the curve C with equation $y = \cosh x$ between $x = -1$ and $x = 1$ where distances are measured in metres.

(i) Sketch the curve C, and compute its length, giving your answer to the nearest cm.

(ii) A solid of revolution is created by rotating C through 2π about the x-axis. Let its volume be V cubic metres and the area of its surface be S square metres. Evaluate both S and V and show that

$$S = 2V\,.$$

[The curved surface area of the solid of revolution formed by rotating the area under the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is given

$$S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\,,$$

and the length of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by

$$l = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\,.]$$

[16 marks]

14. The framework supporting a bridge contains two thin linear struts AB and CD where the coordinates of the points A, B, C and D are given by $A(-2, 1, 6)$, $B(4, 5, 2)$, $C(1, 0, 7)$, $D(5, 4, 5)$.

    Find vector equations for the straight lines AB and CD.

    Let the feet of the common perpendicular between AB and CD be E and F respectively.

    Show that the perpendicular distance between AB and CD is 3 units.

    Solve the equation

$$\mathbf{a} + \lambda_E \mathbf{u} + 3\frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \mathbf{c} + \mu_F \mathbf{v},$$

    for λ_E and μ_F and hence find the coordinates of E and F.

    [You may assume that if the vector equations for two non-parallel straight lines L_1 and L_2 are respectively:

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}, \qquad \text{and} \qquad \mathbf{r}' = \mathbf{c} + \mu \mathbf{v},$$

    where λ and μ are variable scalar parameters, then the perpendicular distance d between L_1 and L_2 is given by

$$d = \frac{|(\mathbf{a} - \mathbf{c}).(\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|}.]$$

[16 marks]

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15. A metal plate is approximately described by the area in the xy-plane enclosed between the curves $y = f(x) = \sqrt{2x - x^2}$ and $y = g(x) = x^3 - 2x^2$ where all distances are measured in metre units.

    Make a rough sketch of the plate.

    Write down (but do not attempt to evaluate) a definite integral representing the area of the plate in units of m^2.

    Use Simpson's rule with 10 subdivisions of the interval $[0, 2]$ to obtain an estimate of the area of the plate to the nearest cm^2.

    [Recall that the weightings of the abscissa in Simpson's rule are $1, 4, 2, 4, ...2, 4, 1$]
    [16 marks]