MATH195 MATHEMATICS 1 FOR CIVIL ENGINEERS

JANUARY 2002

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 52% of the available marks.

SECTION A

1. Write as a single logarithm:

$$\ln(10) - \ln(6) + \ln(18) - \ln(15).$$

[3 marks]

2. The function f is defined by

$$f(x) = \frac{3x+4}{x-5}, \qquad x \neq 5.$$

Find the inverse function $f^{-1}(x)$ and verify that $f^{-1}[f(x)] = x$. [4 marks]

3. The three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors (relative to the origin O) of the points A, B and C with co-ordinates $A(1,-2,3)$, $B(4,-1,2)$ and $C(1,1,3)$. Calculate

(i) $\mathbf{a}.\mathbf{b}$, (ii) $\mathbf{b} \times \mathbf{c}$, (iii) $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$.

Find also the angle between $\mathbf{a}$ and $\mathbf{b}$. [8 marks]

4. Write down the equation of the straight line through the points $(0,-3,-2)$ and $(2,0,4)$. Compute the perpendicular distance of this straight line from the origin. [6 marks]

5. State L'Hôpital's rule for the evaluation of limits. Hence or otherwise evaluate

$$\lim_{x \to 0} \frac{\cos x - \cos 3x}{1 - \cosh 2x}.$$

[6 marks]

6. Differentiate the following functions with respect to x:

(i) $\sinh^5 x$, (ii) $e^{\cos x}$, (iii) $\dfrac{x-2}{x^2+1}$, (iv) $\cos(2x^3)$.

[6 marks]

7. Given that $x^4 e^y + x^3 y^2 = \sin 2y$, find $\frac{dy}{dx}$ as a function of x and y. [4 marks]

8. Evaluate the following integrals:

(i)

$$\int_0^1 xe^{2x}\,dx,$$

[4 marks]

(ii)

$$\int_0^1 \frac{x^3}{\sqrt{1+x^4}}\,dx,$$

[6 marks]

(iii)

$$\int_3^5 \frac{x-5}{(x-1)(x+3)}dx.$$

[5 marks]

SECTION B

9. (i) Find and classify all stationary points of the function f defined by

$$f(x) = x - 6 + \frac{4}{x-1}, \qquad x \neq 1.$$

(Note that for full credit, all mathematical working must be shown in detail.) Sketch the graph of $y = f(x)$, showing clearly the turning points, asymptotes and the points at which the graph intersects the x and y axes. [13 marks]

(ii) Find where the curve intersects the straight line $y = x - 7$. [3 marks]

10. (i) A force $\mathbf{F}$ is applied to a rigid body at a point with position vector $\mathbf{r}$ relative to the origin O. Show that the magnitude of the vector $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ gives the magnitude of the turning moment of $\mathbf{F}$ about O and the direction of $\mathbf{M}$ gives the axis of the rotation which $\mathbf{F}$ would produce. [4 marks]

(ii) A rigid body contains the points P_1, P_2, and P_3 with cartesian co-ordinates $P_1(1, 2, -1)$, $P_2(2, 1, -1)$ and $P_3(2, 1, -2)$. Write down the position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ of P_1, P_2, and P_3 respectively. A force $\mathbf{F}_1 = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ is applied at P_1 and a force $\mathbf{F}_2 = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ is applied at P_2. Show that their total turning moment about O is given by

$$(\mathbf{r}_1 \times \mathbf{F}_1) + (\mathbf{r}_2 \times \mathbf{F}_2) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

Additional forces $\mathbf{F}_3 = \lambda(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, applied at P_3, and $\mathbf{F}_4$, applied at O, also act on the body. Assuming that the body is maintained in static equilibrium under the action of all four forces, find the value of λ and show that

$$\mathbf{F}_4 = -\frac{1}{3}(11\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}).$$

[12 marks]

11. (i) Write down the definitions of $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}. Hence prove that

$$1 + 2\sinh^2 x = \cosh 2x, \qquad 2\sinh x \cosh x = \sinh 2x.$$

[8 marks]

(ii) A bridge support has the form of a flat plate in the shape of the area between the curve $y = \sinh x$ and the x-axis between $x = 0$ and $x = 2$. The centre of gravity of the plate is at $(\bar{x}, \bar{y})$. Show that

$$\bar{x} = \frac{2\cosh 2 - \sinh 2}{\cosh 2 - 1}$$

and calculate $\bar{y}$. [8 marks]

[The centre of gravity of a flat plate of uniform density with boundaries given by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is at $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{A}\int_a^b x f(x) dx, \qquad \bar{y} = \frac{1}{2A}\int_a^b [f(x)]^2 dx,$$

where A is the area of the plate.]

12. (i) A pillar of uniform density has the shape of the solid of revolution made by rotating the area under the curve $y = 1 + x^2$ between $x = 0$ and $x = 1$ about the x-axis. Show that its centre of gravity is at $(\frac{5}{8}, 0, 0)$. [4 marks]

(ii) Write down an expression for the area of the curved surface of the pillar. Use Simpson's Rule to obtain an approximate value for this area, by dividing the interval $[0, 1]$ into ten equal parts and working throughout with at least five significant digits. [12 marks]

[The volume of the solid of revolution formed by rotating the area under the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is given by

$$V = \pi \int_a^b y^2 dx$$

and its centroid is at $(\overline{x}, 0, 0)$, where

$$\overline{x} = \frac{\pi}{V}\int_a^b xy^2\, dx.$$

The area A of the curved surface of this solid of revolution is given by

$$A = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.]$$

13. (a) Let $z_1 = 4 - i$ and $z_2 = 5 + 2i$. Compute $z_1 + z_2$, $z_1 z_2$ and $\frac{z_1}{z_2}$, giving your answers in the form $a + bi$, where a, b are real numbers. [5 marks]

(b) Write the complex number $-1+i$ in modulus-argument form. Hence find all complex numbers z which satisfy $z^3 + 1 - i = 0$. Sketch a diagram showing the position of these complex numbers in the complex plane. [11 marks]