MATH195 MATHEMATICS 1 FOR CIVIL ENGINEERS

JANUARY 2002

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 52% of the available marks.

SECTION A

1. Write as a single logarithm:

$$\ln(12) - \ln(14) + \ln(21) - \ln(9).$$

[3 marks]

2. The function f is defined by

$$f(x) = \frac{2x+5}{x-4}, \qquad x \neq 4.$$

Find the inverse function $f^{-1}(x)$ and verify that $f[f^{-1}(x)] = x$. [4 marks]

3. The three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors (relative to the origin O) of the points A, B and C with co-ordinates $A(2, 1, -3)$, $B(1, 1, -2)$ and $C(3, 4, -1)$. Calculate

(i) $\mathbf{a}.\mathbf{b}$, (ii) $\mathbf{b} \times \mathbf{c}$, (iii) $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$.

Find also the angle between $\mathbf{a}$ and $\mathbf{b}$. [8 marks]

4. Write down the equation of the straight line through the points $(1, 1, 0)$ and $(-1, 0, 2)$. Compute the perpendicular distance of this straight line from the origin. [6 marks]

5. State L'Hôpital's rule for the evaluation of limits. Hence or otherwise evaluate

$$\lim_{x \to 0} \frac{x - \sin x}{x^2 - \cos x + \cosh x}.$$

[6 marks]

6. Differentiate the following functions with respect to x:

(i) $\cos^4 x$, (ii) $\ln \cosh x$, (iii) $\dfrac{x+3}{x^2-2}$, (iv) $\sinh(3x^2)$.

[6 marks]

7. Given that $x^3 \cos y + x^2 y^3 = e^{2y}$, find $\frac{dy}{dx}$ as a function of x and y. [4 marks]

8. Evaluate the following integrals:

(i) $\displaystyle\int_0^1 x^4 \ln x \, dx$, (ii) $\displaystyle\int_0^2 x^2 e^{(x^3)} \, dx$, (iii) $\displaystyle\int_4^5 \frac{x+11}{(x-3)(x+4)} dx.$

[11 marks]

9. Evaluate

$$\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-x^2}} dx.$$

[4 marks]

SECTION B

10. Find and classify all stationary points of the function f defined by

$$f(x) = x - 9 + \frac{4}{x-4}, \qquad x \neq 4.$$

(Note that for full credit, all mathematical working must be shown in detail.) Sketch the graph of $y = f(x)$, showing clearly the turning points, asymptotes and the points at which the graph intersects the x and y axes. What is the equation of the tangent to this curve at $x = 3$? [16 marks]

11. The framework supporting a bridge contains two thin linear struts AB and CD,where the cartesian co-ordinates of the points A, B, C, D are given by $A(8,3,1)$, $B(8,4,2)$, $C(2,4,8)$, $D(4,3,8)$. Find vector equations for the straight lines AB and CD. Let the feet of the common perpendicular between AB and CD be E and F respectively. Determine the co-ordinates of E and F, and hence find the perpendicular distance between AB and CD.

[16 marks]

12. Write down the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}. Show that

$$2\sinh x \cosh x = \sinh 2x, \qquad \cosh^2 x + \sinh^2 x = \cosh 2x.$$

Hence derive an expression for $\tanh 2x$ in terms of $\tanh x$.

The inverse hyperbolic tangent function $y = \tanh^{-1} x$ is defined by the equation $\tanh y = x$, where $-1 < x < 1$. Prove that

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$

Hence or otherwise show that

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}.$$

[16 marks]

13. A pillar of uniform density has the shape of the solid of revolution made by rotating the area under the curve $y = e^x$ between $x = 0$ and $x = 1$ about the x-axis. If its centre of gravity is at $(\overline{x}, 0, 0)$, show that

$$\overline{x} = \frac{1}{2}\frac{e^2+1}{e^2-1}.$$

Write down an expression for the area of the curved surface of the pillar. Use Simpson's Rule to obtain an approximate value for this area, by dividing the interval $[0, 1]$ into ten equal parts and working throughout with at least five significant digits.

[The volume of the solid of revolution formed by rotating the area under the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is given by

$$V = \pi \int_a^b y^2 \, dx$$

and its centroid is at $(\overline{x}, 0, 0)$, where

$$\overline{x} = \frac{\pi}{V} \int_a^b xy^2 \, dx.$$

The area A of the curved surface of this solid of revolution is given by

$$A = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.]$$

[16 marks]

14. (a) Let $z_1 = 3 - 2i$ and $z_2 = 4 + 3i$. Compute $z_1 + z_2$, $z_1 z_2$ and $\frac{z_1}{z_2}$, giving your answers in the form $a + bi$, where a, b are real numbers.

(b) Write the complex number $-1 + \sqrt{3}i$ in modulus-argument form. Hence find all complex numbers z which satisfy $z^4 + 1 - \sqrt{3}i = 0$. Sketch a diagram showing the position of these complex numbers in the complex plane.

[16 marks]