MATH195 MATHEMATICS 1 FOR CIVIL ENGINEERS

JANUARY 2000

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 52% of the available marks.

SECTION A

1. Write as a single logarithm:

$$\ln(12) - \ln(15) + \ln(20) - \ln(8).$$

[3 marks]

2. The function f is defined by

$$f(x) = \frac{5x+2}{x-3}, \qquad x \neq 3.$$

Find the inverse function $f^{-1}(x)$ and verify that $f[f^{-1}(x)] = x$. [4 marks]

3. The three vectors **a**, **b** and **c** are the position vectors (relative to the origin O) of the points A, B and C with co-ordinates $A(3,1,2)$, $B(2,4,-3)$ and $C(1,1,-1)$. Calculate

$$\text{(i)} \quad \mathbf{a}.\mathbf{b}, \qquad \text{(ii)} \quad \mathbf{b}\times\mathbf{c}, \qquad \text{(iii)} \quad \mathbf{a}.(\mathbf{b}\times\mathbf{c}).$$

Find also the angle between **a** and **b**. [8 marks]

4. Write down the equation of the straight line through the points $(3,-2,2)$ and $(1,-1,0)$. Compute the perpendicular distance of this straight line from the origin. [6 marks]

5. State L'Hôpital's rule for the evaluation of limits. Hence or otherwise evaluate

$$\lim_{x\to 0} \frac{x\sin x}{1-\cos 2x}.$$

[6 marks]

6. Differentiate the following functions with respect to x:

$$\text{(i)} \quad \sinh(x^4), \qquad \text{(ii)} \quad e^{x\cosh x}, \qquad \text{(iii)} \quad \frac{x-1}{x^2+2}, \qquad \text{(iv)} \quad \cos^5 x.$$

[6 marks]

7. Given that $y^3 \sin x + \cos y = e^{y^2}$, find $\frac{dy}{dx}$ as a function of x and y. [4 marks]

8. Evaluate the following integrals:

(i) $\displaystyle\int_0^2 xe^{2x}\,dx,$ (ii) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x\,dx,$ (iii) $\displaystyle\int_3^5 \frac{3x+4}{(x-2)(x+3)}dx.$

[11 marks]

9. Evaluate

$$\int_0^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}dx$$

using the substitution $x = 2\sin\theta$. [4 marks]

SECTION B

10. Find and classify all stationary points of the function f defined by

$$f(x) = \frac{x^2 - 11x + 28}{x - 8}, \qquad x \neq 8.$$

(Note that for full credit, all mathematical working must be shown in detail.) Sketch the graph of $y = f(x)$, showing clearly the turning points, asymptotes and the points at which the graph intersects the x and y axes. What is the equation of the tangent to this curve at $x = 12$? [16 marks]

11. The framework supporting a bridge contains two thin linear struts AB and CD,where the cartesian co-ordinates of the points A, B, C, D are given by $A(-2, 1, 6)$, $B(4, 5, 2)$, $C(1, 0, 7)$, $D(5, 4, 5)$. Find vector equations for the straight lines AB and CD, and hence calculate the perpendicular distance between AB and CD. Let the feet of the common perpendicular between AB and CD be E and F respectively. Determine the co-ordinates of E and F.

[You may assume that if the vector equations for two non-parallel straight lines L_1 and L_2 are respectively:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}, \qquad \text{and} \qquad \mathbf{r}' = \mathbf{c} + \mu\mathbf{v},$$

where λ and μ are variable scalar parameters, then the perpendicular distance d between L_1 and L_2 is given by

$$d = \frac{|(\mathbf{a} - \mathbf{c}).(\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|}.]$$

[16 marks]

12. Write down the definitions of $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}. Hence prove that

$$2\cosh^2 x - 1 = \cosh 2x, \qquad 2\sinh x \cosh x = \sinh 2x.$$

A bridge support has the form of a flat plate in the shape of the area between the curve $y = \cosh x$ and the x-axis between $x = 0$ and $x = 2$. Calculate the position of the centre of gravity of the plate.

[The centre of gravity of a flat plate of uniform density with boundaries given by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is at $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{A}\int x f(x)dx, \qquad \bar{y} = \frac{1}{2A}\int [f(x)]^2 dx,$$

where A is the area of the plate.] [16 marks]

13. A loudspeaker is made by rotating the part of the curve $y = \frac{1}{3}x^3$ between $x = 1$ and $x = \sqrt{2}$ about the x-axis. Show that the area A of the curved surface of the loudspeaker is given by

$$A = \tfrac{\pi}{9}(5\sqrt{5} - 2\sqrt{2}).$$

Write down an expression for the length of the curve $y = \frac{1}{3}x^3$ between $x = 0$ and $x = 1$. Use Simpson's Rule to obtain an approximate value for this length, by dividing the interval $[0, 1]$ into ten equal parts and working throughout with at least five significant digits.

[The area S of the curved surface formed by rotating the part of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is given by

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

and the length l of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.]$$

[16 marks]

14. (a) Let $z_1 = 2 - i$ and $z_2 = 3 + 4i$. Compute $z_1 + z_2$, $z_1 z_2$ and $\frac{z_1}{z_2}$, giving your answers in the form $a + bi$, where a, b are real numbers.

(b) Write the complex number $-1 + i$ in modulus-argument form. Hence find all complex numbers z which satisfy $z^3 + 1 = i$. Sketch a diagram showing the position of these complex numbers in the complex plane. [16 marks]