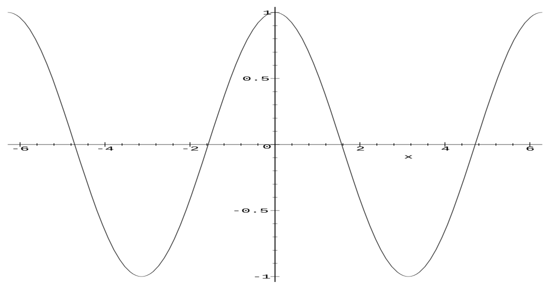
MATH191 Exam September 2004, Solutions

1. The maximal domain is $\mathbb{R}$ and the range is $[-1, 1]$.

The graph is shown below. It crosses the y-axis at $y = 1$, and the x-axis at $x = -3\pi/2, -\pi/2, \pi/2, 3\pi/2$.



2. We have $f(0) = 0$, $f'(x) = -2/(1-2x)$, so $f'(0) = -2$, and $f''(x) = -4/(1-2x)^2$, so $f''(0) = -4$.

Hence the Maclaurin series expansion of $f(x)$ up to the term in x^2 is

$$f(x) = -2x - 2x^2 + \cdots$$

3.

a) $r = \sqrt{1+1} = \sqrt{2}$. $\tan\theta = 1/-1 = -1$, so since $x < 0$ we have $\theta = \tan^{-1}(-1) + \pi = 3\pi/4$.

b) $x = 3\cos(2\pi/3) = 3(-1/2) = -3/2$. $y = 3\sin(2\pi/3) = 3\sqrt{3}/2$.

4.

$$\begin{aligned}\int_{-1}^{1} \left(e^{3x} - \cosh x\right)\, dx &= \left[\frac{e^{3x}}{3} - \sinh x\right]_{-1}^{1} \\ &= e^3/3 - \sinh(1) - e^{-3}/3 + \sinh(-1) = 2\sinh(3)/3 - 2\sinh(1) = 4.328\end{aligned}$$

to three decimal places.

5. Differentiate the defining equation with respect to x to obtain

$$4x - 2y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} + 2 = 0,$$

giving

$$\frac{dy}{dx} = \frac{4x - 3y + 2}{3x + 2y}.$$

At $(x, y) = (1, 1)$ this gives $\frac{dy}{dx} = \frac{3}{5}$.
Hence the equation of the tangent is

$$y = 1 + 3(x - 1)/5 \qquad \text{or} \qquad 5y = 3x + 2.$$

6.

a) By the product rule,

$$\frac{d}{dx}\left(x^2 \cosh x\right) = 2x \cosh x + x^2 \sinh x.$$

b) By the chain rule,

$$\frac{d}{dx}(x + \sin x)^4 = 4(x + \sin x)^3(1 + \cos x).$$

c) By the quotient rule,

$$\frac{d}{dx}\left(\frac{x^2}{1 + e^x}\right) = \frac{2x(1 + e^x) - x^2 e^x}{(1 + e^x)^2}.$$

7. $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$. Stationary points are given by solutions of $f'(x) = 0$, so there are exactly two stationary points, namely $x = 1$ and $x = 2$.

To determine their natures, $f''(x) = 6(2x - 3)$ so $f''(1) < 0$ and $f''(2) > 0$. Hence $x = 1$ is a local maximum and $x = 2$ is a local minimum.

8.

$$\begin{aligned} z_1 + z_2 &= 3 + 2j \\ z_1 - z_2 &= 1 - 4j \\ z_1 z_2 &= (2 - j)(1 + 3j) = 2 + 6j - j - 3j^2 = 5 + 5j \\ z_1/z_2 &= \frac{(2 - j)(1 - 3j)}{(1 + 3j)(1 - 3j)} = \frac{-1 - 7j}{10} \end{aligned}$$

9. $\sin^{-1}(1/2) = \pi/6$.
Hence the general solution of $\sin\theta = 1/2$ is

$$\theta = \begin{cases} \frac{\pi}{6} + 2n\pi & (n \in \mathbb{Z}) \\ \frac{5\pi}{6} + 2n\pi & (n \in \mathbb{Z}) \end{cases}$$

10.

$$\begin{aligned} \mathbf{a}+\mathbf{b} &= 4\mathbf{i}+\mathbf{k} \\ \mathbf{a}-\mathbf{b} &= 2\mathbf{i}+2\mathbf{j}-3\mathbf{k} \\ |\mathbf{a}| &= \sqrt{3^2+1^2+1^2}=\sqrt{11} \\ |\mathbf{b}| &= \sqrt{1^2+1^2+2^2}=\sqrt{6} \\ \mathbf{a}\cdot\mathbf{b} &= 3-1-2=0 \end{aligned}$$

Hence the angle between **a** and **b** is $\pi/2$.

11. The Maclaurin series expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Hence

a)

$$x\sin x = x^2 - \frac{x^4}{3!} + \cdots$$

b)

$$\sin(2x) = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \cdots = 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \cdots$$

c)

$$\begin{aligned} \sin^2 x &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\ &= x^2 - \frac{x^4}{3} + \cdots \end{aligned}$$

c) gives an approximation of $(0.1)^2 - (0.1)^4/3 = 0.01 - 0.0000333 = 0.0099667$ to $\sin^2(0.1)$ (6 decimal places).

12. The radius of the convergence R of the power series

$$\sum_{n=1}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|,$$

provided this limit exists.

In this case $a_n = 1/n^2 2^n$, so $|a_n/a_{n+1}| = (n+1)^2 2^{n+1}/(n^2 2^n) = 2((n+1)/n))^2$, which tends to 2 as $n \to \infty$. Hence $R = 2$.

When $x = -2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

This converges by the alternating series test, which states that

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges if a_n is a decreasing sequence with $a_n \to 0$.

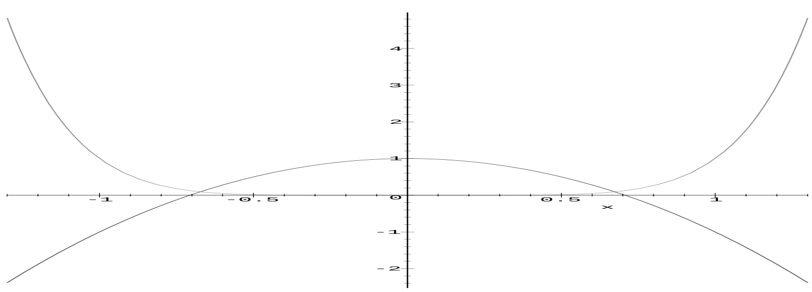
When $x = 2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges (standard result).

Hence the series converges if and only if $-2 \le x \le 2$.

13. The graphs are as shown:



In $[0, \infty)$, $1 - 2x^2$ decreases strictly from 1 to $-\infty$, and x^6 increases strictly from 0 to ∞: it follows that there is exactly one solution in $x > 0$. By symmetry, there is exactly one solution in $x < 0$.

Setting $f(x) = x^6 + 2x^2 - 1$, we have $f'(x) = 6x^5 + 4x$, so the Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^6 + 2x_n^2 - 1}{6x_n^5 + 4x_n}.$$

Hence

$$x_1 = x_0 - \frac{x_0^6 + 2x_0^2 - 1}{6x_0^5 + 4x_0} = 0.674360.$$

$$x_2 = x_1 - \frac{x_1^6 + 2x_1^2 - 1}{6x_1^5 + 4x_1} = 0.673350.$$

$$x_3 = x_2 - \frac{x_2^6 + 2x_2^2 - 1}{6x_2^5 + 4x_2} = 0.673348.$$

By symmetry, the best approximation available to the second solution of $f(x) = 0$ is $x = -0.673348$.

14. For $x \leq 0$ we have $f(x) = x^2 + 2x - 1$, which has zeros at $x = -1 \pm \sqrt{2}$, of which only $-1-\sqrt{2}$ lies in the appropriate domain. The derivative is $f'(x) = 2x+2$, so there is a stationary point at $x = -1$. Since $f''(x) = 2$, the stationary point is a local minimum. $f(x) = 1 - 2 - 1 = -2$ at the stationary point. The gradient of $x^2 + 2x - 1$ at $x = 0$ is 2.

For $x < 0$ we have $f(x) = 2/(x-2)$, which has no zeros and tends to -1 as $x \to 0$ and to 0 as $x \to \infty$. $f'(x) = -2/(x-2)^2$, so there are no stationary points, and $f(x)$ is decreasing in $(0, 2) \cup (2, \infty)$. The gradient is $-1/2$ at $x = 0$. There is a vertical asymptote at $x = 2$.

The graph of $f(x)$ is therefore

$f(x)$ is not continuous at $x = 2$, since 2 is not in its maximal domain.

$f(x)$ is not differentiable at $x = 2$ (not in maximal domain), or at $x = 0$ (since the pieces join with different gradients).

15. Let $z = \cos\theta + j\sin\theta$, so by de Moivre's theorem

$$\begin{aligned} z^n &= \cos n\theta + j\sin n\theta \\ z^{-n} &= \cos n\theta - j\sin n\theta. \end{aligned}$$

Thus $z^n - z^{-n} = 2j\sin n\theta$.

In particular, $2j\sin\theta = z - z^{-1}$ so

$$\begin{aligned} 8j^3\sin^3\theta &= (z - z^{-1})^3 \\ &= z^3 - 3z + 3z^{-1} - z^{-3} \\ &= (z^3 - z^{-3}) - 3(z - z^{-1}) \\ &= 2j\sin 3\theta - 6j\sin\theta. \end{aligned}$$

Thus, since $j^3 = -j$,

$$4\sin^3\theta = -\sin 3\theta + 3\sin\theta,$$

so $a = -1$ and $b = 3$.

So

$$\begin{aligned} \int_0^{\pi/2} \sin^3 x\,dx &= \frac{1}{4}\int_0^{\pi/2} (3\sin\,x - \sin(3x))\,dx \\ &= \frac{1}{4}\left[\frac{\cos(3x)}{3} - 3\cos\,x\right]_0^{\pi/2} \\ &= \frac{1}{4}\left(\frac{\cos(3\pi/2) - \cos(0)}{3} - 3(\cos(\pi/2) - \cos(0))\right) \\ &= \frac{1}{4}\left(\frac{-1}{3} + 3\right) = 2/3. \end{aligned}$$