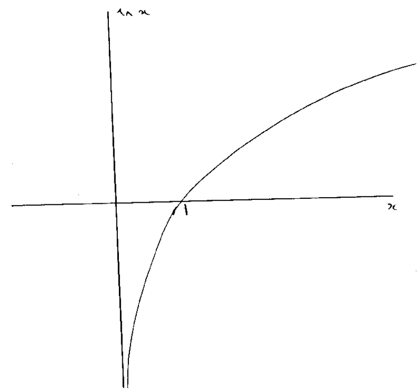
MATH191 Exam September 2002, Solutions

1. The maximal domain is $(0, \infty)$.
The graph is shown below.



2. We have $f(0) = \ln(1) = 0$, $f'(x) = 3(1 + 3x)^{-1}$, so $f'(0) = 3$, and $f''(x) = -9(1 + 3x)^{-2}$, so $f''(0) = -9$.

Hence the Maclaurin series expansion of $f(x)$ up to the term in x^2 is

$$f(x) = 3x - \frac{9}{2}x^2 + \cdots.$$

3.

a) This is a polynomial with only even powers, and hence is even.

b) Let $f(x) = x/(1+x^2)$. Then $f(-x) = (-x)/(1+(-x)^2) = -x/(1+x^2) = -f(x)$. Hence $f(x)$ is odd.

c) Let $f(x) = x \sinh x$. Then $f(-x) = (-x)\sinh(-x) = (-x)(-\sinh x) = x \sinh x = f(x)$. Hence $f(x)$ is even.

4.

$$\begin{aligned} \int_1^2 e^x + x^{-2}\,dx &= \left[e^x - x^{-1}\right]_1^2 \\ &= e^2 - e^1 - \frac{1}{2} + 1 \\ &= 5.171 \end{aligned}$$

to three decimal places.

5.

a) $(x^2-2x-3)/(x-3)=x+1$, provided $x\neq 3$. Hence the limit exists and is equal to 4.

b) By L'Hôpital's rule,

$$\lim_{x\to 0}\frac{\sin(2x)}{x}=\lim_{x\to 0}\frac{2\cos(2x)}{1}.$$

Hence the limit exists and is equal to 2.

c) Since $-1\leq \sin(2x)\leq 1$ for all x, we have $-1/x\leq \sin(2x)/x\leq 1/x$ for $x>0$, and hence $\sin(2x)/x\to 0$ as $x\to\infty$.

6.

a) By the product rule,

$$\frac{d}{dx}e^x\cos x=e^x\cos x-e^x\sin x=e^x(\cos x-\sin x).$$

b) By the chain rule,

$$\frac{d}{dx}\ln(1+x^4)=\frac{4x^3}{1+x^4}.$$

c) By the quotient rule,

$$\frac{d}{dx}\frac{\sin x}{1+x^2}=\frac{(1+x^2)\cos x-2x\sin x}{(1+x^2)^2}.$$

7. $f'(x)=\frac{1}{x}-4x$. Stationary points are given by solutions of $f'(x)=0$, or $x^2=1/4$, which has two solutions, $x=\pm 1/2$. However, $-1/2$ is not in the maximal domain of $f(x)$, so there is just one stationary point, at $x=1/2$.

To determine its nature, note that $f''(x)=\frac{-1}{x^2}-4$ is negative when $x=1/2$. Hence the stationary point is a maximum.

8.

$$\begin{aligned} z_1+z_2 &= 4+3j\\ z_1-z_2 &= 2-j\\ z_1z_2 &= 1+7j\\ z_1/z_2 &= \frac{(3+j)(1-2j)}{(1+2j)(1-2j)}=\frac{5-5j}{5}=1-j. \end{aligned}$$

9. $\tan^{-1}(1) = \pi/4$.

Hence the general solution of $\tan\theta = 1$ is

$$\theta = \frac{\pi}{4} + n\pi \qquad (n \in \mathbb{Z}).$$

10.

$$\begin{aligned} \mathbf{a}+\mathbf{b} &= 4\mathbf{i} - 2\mathbf{j} \\ \mathbf{a}-\mathbf{b} &= -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \\ |\mathbf{a}| &= \sqrt{1^2+2^2+1^2} = \sqrt{6} \\ |\mathbf{b}| &= \sqrt{3^2+0^2+1^2} = \sqrt{10} \\ \mathbf{a}\cdot\mathbf{b} &= 3+0-1 = 2. \end{aligned}$$

Hence the angle between **a** and **b** is $\cos^{-1}(2/\sqrt{60}) = 1.310$ to 3 decimal places.

11. The Maclaurin series expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Hence

a)

$$x\sin x = x^2 - \frac{x^4}{3!} + \cdots$$

b)

$$\sin(2x) = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \cdots = 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \cdots$$

c)

$$\begin{aligned} \sin^2 x &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\ &= x^2 - \frac{x^4}{3} + \cdots. \end{aligned}$$

c) gives an approximation of $(0.1)^2 - (0.1)^4/3 = 0.01 - 0.0000333 = 0.0099667$ to $\sin^2(0.1)$ (6 decimal places).

12. The radius of the convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case $a_n = (-1)^n/(n2^n)$, so $|a_n/a_{n+1}| = 2(n+1)/n = 2 + 2/n$, which tends to 2 as $n \to \infty$. Hence $R = 2$.

This means that the series converges for $-2 < x < 2$, and diverges for $x > 2$ or $x < -2$.

When $x = 2$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}.$$

This converges by the alternating series test, which states that

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges if a_n is a decreasing sequence with $a_n \to 0$.

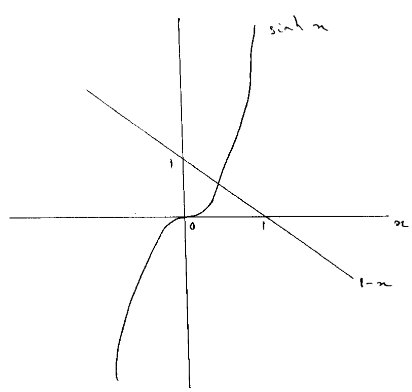
When $x = -2$, the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n},$$

which is divergent (standard result).

Hence the series converges precisely when $-2 < x \leq 2$.

13. The graphs are as shown:



Since $\sinh x$ is increasing and $1 - x$ is decreasing, there can be at most one solution to $\sinh x = 1 - x$. Since the functions take values 0 and 1 at $x = 0$, and values $\sinh(1) > 0$ and 0 at $x = 1$, there is a solution in $[0, 1]$.

Setting $f(x) = \sinh\, x + x - 1$, we have $f'(x) = \cosh\, x + 1$, so the Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{\sinh\, x_n + x_n - 1}{\cosh\, x_n + 1}.$$

Hence

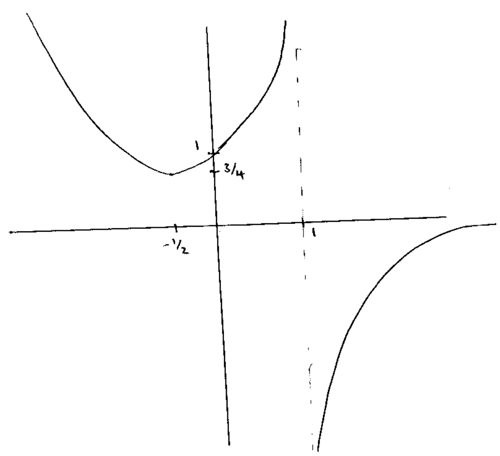
$$x_1 = 0.493357, \quad x_2 = 0.490074, \quad \text{and } x_3 = 0.490073$$

to 6 decimal places.

14. For $x \leq 0$ we have $f(x) = x^2 + x + 1$, which has no real zeros. The derivative is $f'(x) = 2x + 1$, so there is a stationary point at $x = -1/2$. Since $f''(x) = 2$, the stationary point is a local minimum. $f(x) = \frac{3}{4}$ at the stationary point. The gradient of $x^2 + x + 1$ at $x = 0$ is 1.

For $x > 0$ we have $f(x) = 1/(1-x)$, which has no zeros and tends to 1 as $x = 0$, and to 0 as $x \to \infty$. $f'(x) = 1/(1-x)^2$, so there are no stationary points, and $f(x)$ is increasing in $(0,1) \cup (1,\infty)$; the gradient is 1 at $x = 0$. There is a vertical asymptote at $x = 1$.

The graph of $f(x)$ is therefore



$f(x)$ is not continuous at $x = 1$, since 1 is not in its maximal domain.

$f(x)$ is not differentiable at $x = 1$ (not in maximal domain): however, it is differentiable with derivative 1 at $x = 0$.

15. Let $z = \cos\theta + j\sin\theta$, so by de Moivre's theorem

$$\begin{aligned} z^n &= \cos n\theta + j\sin n\theta \\ z^{-n} &= \cos n\theta - j\sin n\theta. \end{aligned}$$

Thus $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2j\sin n\theta$.

In particular, $2\cos\theta = z + z^{-1}$ so

$$\begin{aligned}
16\cos^4\theta &= (z+z^{-1})^4 \\
&= z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4} \\
&= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6 \\
&= 2\cos 4\theta + 8\cos 2\theta + 6.
\end{aligned}$$

Thus

$$8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3,$$

so $a = 1$, $b = 4$, and $c = 3$.

Similarly, $2j\sin\theta = z - z^{-1}$, so

$$\begin{aligned}
16\sin^4\theta &= (z-z^{-1})^4 \\
&= z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4} \\
&= (z^4 + z^{-4}) - 4(z^2 + z^{-2}) + 6 \\
&= 2\cos 4\theta - 8\cos 2\theta + 6.
\end{aligned}$$

Thus

$$8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3,$$

so $d = 1$, $e = -4$, and $f = 3$.

When $\theta = \pi/2$, $\sin\theta = 1$, $\cos\theta = 0$, $\cos\, 2\theta = -1$, and $\cos\, 4\theta = 1$.

Thus the identity for $8\cos^4\theta$ reads $0 = 1 - 4 + 3$; and the identity for $8\sin^4\theta$ reads $8 = 1 - (-4) + 3$: so both check.