SECTION A

1. State the maximal domain of the function

$$f(x) = \sin x.$$

Sketch the graph of $y = f(x)$ for $-2\pi \leq x \leq 2\pi$, indicating the values of x where it crosses the x-axis.

[4 marks]

2. By evaluating $f(0)$, $f'(0)$, and $f''(0)$, obtain the Maclaurin series expansion of the function

$$f(x) = \ln(1 - 2x)$$

up to and including the term in x^2.

[5 marks]

3. State, with reasons, whether the following functions are odd, even, or neither:

$$\text{a) } x^3 + 1; \qquad \text{b) } \frac{1 + x^2}{x}; \qquad \text{c) } e^x \sin x.$$

[6 marks]

4. Calculate the integral

$$\int_0^{\pi/2} x + \sin(2x)\,\mathrm{d}x,$$

evaluating the result to three decimal places.

[5 marks]

5. Which of the following limits exist? Evaluate those which do.

$$\text{a) } \lim_{x\to 2} \frac{x^2-3x+2}{x-2}; \qquad \text{b) } \lim_{x\to 0} \frac{\cos x - 1}{x^2}; \qquad \text{c) } \lim_{x\to\infty} \cos(2x).$$

[8 marks]

6. Differentiate the following functions:

$$\text{a) } x\cos x; \qquad \text{b) } e^{(-x^2)}; \qquad \text{c) } \frac{x}{1-x^2}.$$

[6 marks]

7. Show that the function

$$f(x) = x^3 - 3x + 5$$

has exactly two stationary points. Determine whether each of these stationary point is a local maximum, a local minimum, or a point of inflection.

[5 marks]

8. Consider the curve defined by

$$x^3 + x^2y + y^3 = 1.$$

Find an expression for $\frac{dy}{dx}$ in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (1, 0)$.

[6 marks]

9. Use your calculator to find a solution (in radians, to three decimal places) of the equation

$$\cos\theta = \frac{1}{4}.$$

What is the general solution of this equation?

[4 marks]

10. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

[6 marks]

SECTION B

11. Give the Maclaurin series expansion of the function $f(x) = \cos x$ up to and including the term in x^4 (you are not required to show any working if you remember this expansion).

[2 marks]

Hence, or otherwise, determine the Maclaurin series expansions of the following functions, up to and including the terms in x^4:

a) $x \cos x$; b) $\cos(2x)$; c) $\cos^2 x$.

[10 marks]

Use your answer to c) to obtain an approximation to $\cos^2(0.1)$. Give your approximation to 6 decimal places.

[3 marks]

12. Calculate the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \frac{n-1}{n\,3^n}\, x^n.$$

[8 marks]

Write down the series when $x = R$ and when $x = -R$, and state whether it is convergent or divergent in each case. Hence state all of the (real) values of x for which the power series converges.

[7 marks]

13. By sketching the graphs of $y = x^3 + 1$ and $y = x^2$ on the same axes, explain why the equation

$$f(x) = x^3 + 1 - x^2 = 0$$

has exactly one solution with $x \leq 0$ (you are not required to consider whether there are any solutions with $x > 0$).

[7 marks]

Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = -0.8$ to obtain successive approximations x_1, x_2, and x_3 to this solution. You should give each approximation to 6 decimal places.

[8 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 + x & \text{if } x \leq 0 \\ \frac{x}{x-2} & \text{if } x > 0. \end{cases}$$

Sketch the graph of $y = f(x)$, indicating clearly the positions of any zeros, stationary points, and asymptotes. (You will get no marks for sketching the graph unless you show how you have determined the positions of these features.)

[12 marks]

At which value or values of x is $f(x)$ not continuous? At which value or values is it not differentiable?

[3 marks]

15.

By using de Moivre's theorem, find the integers a, b, c, d, e, and f such that

$$\begin{aligned} \cos(4\theta) &= a\cos^4\theta + b\cos^2\theta + c \qquad \text{and} \\ \sin(4\theta) &= 4\tan\theta\left(d\sin^4\theta + e\sin^2\theta + f\right) \end{aligned}$$

[11 marks]

Check both of these results when $\theta = \frac{\pi}{4}$ (you should work with exact values of $\cos\theta$ and $\sin\theta$, rather than evaluating them on your calculator).

[4 marks]