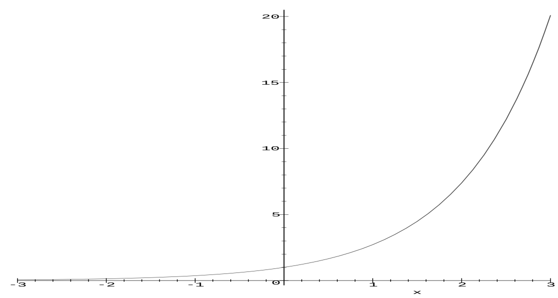
MATH191 Exam January 2004, Solutions

1. The maximal domain is $\mathbb{R}$ and the range is $(0, \infty)$.

The graph is shown below. It crosses the y-axis at $y = 1$.



2. We have $f(0) = 1$, $f'(x) = -(1 + 2x)^{-3/2}$, so $f'(0) = -1$, and $f''(x) = 3(1 + 2x)^{-5/2}$, so $f''(0) = 3$.

Hence the Maclaurin series expansion of $f(x)$ up to the term in x^2 is

$$f(x) = 1 - x + \frac{3x^2}{2} + \cdots$$

3.

a) $r = \sqrt{1+1} = \sqrt{2}$. $\tan\theta = -1/-1 = 1$, so since $x < 0$ we have $\theta = \tan^{-1}(1) + \pi = 5\pi/4$.

b) $x = 2\cos(-\pi/3) = 2(1/2) = 1$. $y = 2\sin(-\pi/3) = 2(-\sqrt{3}/2) = -\sqrt{3}$.

4.

$$\begin{aligned} \int_1^3 \left(\frac{1}{x} + \sin(2x)\right) dx &= \left[\ln(|x|) - \frac{\cos(2x)}{2}\right]_1^3 \\ &= \ln(3) - \ln(1) - \cos(6)/2 + \cos(2)/2 = 0.410 \end{aligned}$$

to three decimal places.

At $(x, y) = (4, 2)$ this gives $\frac{dy}{dx} = \frac{2}{8} = \frac{1}{4}$.

Hence the equation of the tangent is

$$y = 2 + (x - 4)/4 \qquad \text{or} \qquad y = (x + 4)/4.$$

5.

a) By the product rule,

$$\frac{d}{dx}\left((1-x)e^x\right) = -e^x + (1-x)e^x = -xe^x.$$

b) By the chain rule,

$$\frac{d}{dx}(x^2+x+1)^5 = 5(2x+1)(x^2+x+1)^4.$$

c) By the quotient rule,

$$\frac{d}{dx}\left(\frac{\sin x}{x^2+1}\right) = \frac{(x^2+1)\cos x - 2x\sin x}{(x^2+1)^2}.$$

6. $f'(x) = \frac{1}{x} - 2x$. Stationary points are given by solutions of $f'(x) = 0$, or $2x^2 = 1$, so there is exactly one stationary point, namely $x = 1/\sqrt{2}$.

To determine its nature, $f''(x) = -1/x^2 - 2$ so $f''(1/\sqrt{2}) < 0$, and $1/\sqrt{2}$ is a local maximum.

7.

$$\begin{aligned}
z_1 + z_2 &= 2 + 3j \\
z_1 - z_2 &= 4 + j \\
z_1 z_2 &= (3+2j)(-1+j) = -3 - 2j + 3j + 2j^2 = -5 + j \\
z_1/z_2 &= \frac{(3+2j)(-1-j)}{(-1+j)(-1-j)} = \frac{-1-5j}{2}.
\end{aligned}$$

8. $\cos^{-1}(\sqrt{3}/2) = \pi/6$.

Hence the general solution of $\cos\theta = \sqrt{3}/2$ is

$$\theta = \pm\frac{\pi}{6} + 2n\pi \qquad (n \in \mathbb{Z}).$$

9.

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} \\
\mathbf{a} - \mathbf{b} &= -\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} \\
|\mathbf{a}| &= \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3 \\
|\mathbf{b}| &= \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \\
\mathbf{a} \cdot \mathbf{b} &= 6 + 0 - 4 = 2.
\end{aligned}$$

Hence the angle between **a** and **b** is $\cos^{-1}(\frac{2}{15}) = 1.437$ to 3 decimal places.

10. The Maclaurin series expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

Hence

a)

$$xe^x = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \cdots$$

b)

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \cdots$$

c) $(e^x)^2 = e^{2x}$, so the expansion is as in b).

d)

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \cdots$$

The approximation is $1 + \frac{1}{10} + \frac{1}{200} + \frac{1}{6000} + \frac{1}{240000} = 1.105171$ to 6 decimal places.

11. The radius of the convergence R of the power series

$$\sum_{n=1}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists.

In this case $a_n = (-1)^n/(\sqrt{n}3^n)$, so $|a_n/a_{n+1}| = (\sqrt{n+1}3^{n+1})/(\sqrt{n}3^n) = 3\sqrt{n+1}/\sqrt{n}$, which tends to 3 as $n \to \infty$. Hence $R = 3$.

When $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This converges by the alternating series test, which states that

$$\sum_{n=1}^{\infty}(-1)^n a_n$$

converges if a_n is a decreasing sequence with $a_n \to 0$.

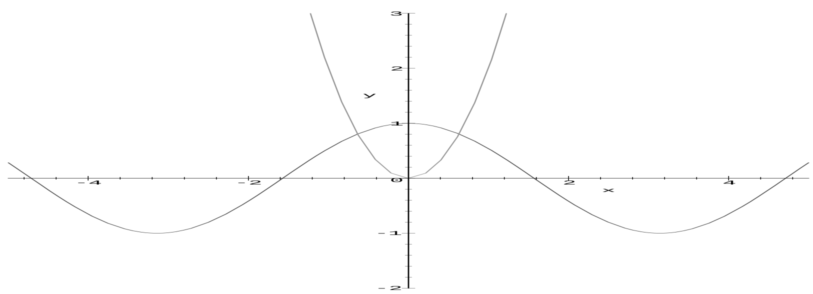
When $x = -3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which diverges (by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$, whose divergence is a standard result).

Hence the series converges if and only if $-3 < x \leq 3$.

12. The graphs are as shown:



In $[0, \pi/2]$, $2x^2$ increases from 0 to $\pi^2/2 > 1$, and $\cos x$ decreases from 1 to 0: it follows that there is exactly one solution between 0 and $\pi/2$. Since $2x^2 > 1$ for $x > \pi/2$, there are no solutions for $x > \pi/2$. Hence there is exactly one solution in $x > 0$. By symmetry, there is exactly one solution in $x < 0$.

Setting $f(x) = 2x^2 - \cos x$, we have $f'(x) = 4x + \sin x$, so the Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{2x_n^2 - \cos x_n}{4x_n + \sin x_n}.$$

Hence

$$x_1 = x_0 - \frac{2x_0^2 - \cos x_0}{4x_0 + \sin x_0} = 0.635531.$$

$$x_2 = x_1 - \frac{2x_1^2 - \cos x_1}{4x_1 + \sin x_1} = 0.634561.$$

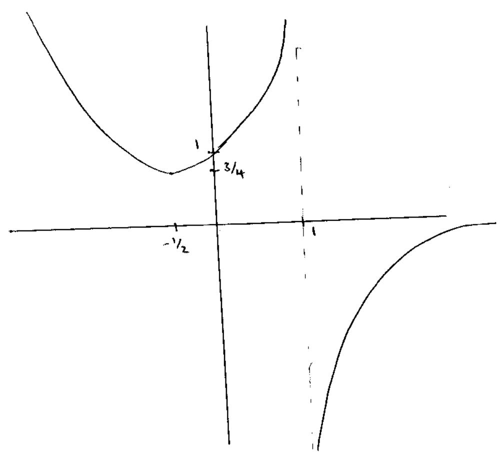
$$x_3 = x_2 - \frac{2x_2^2 - \cos x_2}{4x_2 + \sin x_2} = 0.634560.$$

By symmetry, the best approximation available to the second solution of $f(x) = 0$ is $x = -0.634560$.

13. For $x \leq 0$ we have $f(x) = x^2 + x + 1$, which has no real zeros. The derivative is $f'(x) = 2x + 1$, so there is a stationary point at $x = -1/2$. Since $f''(x) = 2$, the stationary point is a local minimum. $f(x) = \frac{3}{4}$ at the stationary point. The gradient of $x^2 + x + 1$ at $x = 0$ is 1.

For $x > 0$ we have $f(x) = 1/(1 - x)$, which has no zeros and tends to 1 as $x = 0$, and to 0 as $x \to \infty$. $f'(x) = 1/(1-x)^2$, so there are no stationary points, and $f(x)$ is increasing in $(0, 1) \cup (1, \infty)$; the gradient is 1 at $x = 0$. There is a vertical asymptote at $x = 1$.

The graph of $f(x)$ is therefore



$f(x)$ is not continuous at $x = 1$, since 1 is not in its maximal domain.

$f(x)$ is not differentiable at $x = 1$ (not in maximal domain): however, it is differentiable with derivative 1 at $x = 0$.

14. Let $z = \cos\theta + j\sin\theta$, so by de Moivre's theorem

$$\begin{aligned} z^n &= \cos n\theta + j\sin n\theta \\ z^{-n} &= \cos n\theta - j\sin n\theta. \end{aligned}$$

Thus $z^n + z^{-n} = 2\cos n\theta$.

In particular, $2\cos\theta = z + z^{-1}$ so

$$\begin{aligned} 16\cos^4\theta &= (z + z^{-1})^4 \\ &= z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4} \\ &= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6 \\ &= 2\cos 4\theta + 8\cos 2\theta + 6. \end{aligned}$$

Thus

$$8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3,$$

so $a = 1$, $b = 4$, and $c = 3$.

So

$$\begin{aligned}
\int_0^{\pi/6} \cos^4 x\,dx &= \frac{1}{8}\int_0^{\pi/2} \cos(4x) + 4\cos(2x) + 3\,dx \\
&= \frac{1}{8}\left[\frac{\sin(4x)}{4} + 2\sin(2x) + 3x\right]_0^{\pi/6} \\
&= \frac{1}{8}\left(\frac{\sin(2\pi/3)}{4} + 2\sin(\pi/3) + \pi/2\right) \\
&= \frac{1}{8}\left(\frac{\sqrt{3}}{8} + \sqrt{3} + \pi/2\right) = 9\sqrt{3}/64 + \pi/16.
\end{aligned}$$

(A solution in decimals, 0.439919, is acceptable.)