SECTION A

1. State the maximal domain and range of the function

$$f(x) = e^x.$$

Sketch the graph of $y = f(x)$, indicating the coordinates of any points where the graph crosses the axes.

[4 marks]

2. By evaluating $f(0)$, $f'(0)$, and $f''(0)$, obtain the Maclaurin series expansion of the function

$$f(x) = \frac{1}{\sqrt{1+2x}}$$

up to and including the term in x^2.

[5 marks]

3.

a) Convert $(x, y) = (-1, -1)$ from Cartesian to polar coordinates.

b) Convert $(r, \theta) = (2, -\pi/3)$ from polar to Cartesian coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{2}$, etc.

[6 marks]

4. Calculate the integral

$$\int_1^3 \left(\frac{1}{x} + \sin(2x)\right) \mathrm{d}x,$$

evaluating the result to three decimal places.

[5 marks]

5. Consider the curve defined by

$$x^2 + y^2 - 3xy = -4.$$

Find an expression for $\frac{dy}{dx}$ in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (4, 2)$. [8 marks]

6. Differentiate the following functions with respect to x:

$$\text{a) } (1-x)e^x; \qquad \text{b) } (x^2+x+1)^5; \qquad \text{c) } \frac{\sin x}{x^2+1}.$$

[6 marks]

7. Show that the function

$$f(x) = \ln x - x^2 \qquad (x > 0)$$

has exactly one stationary point. Determine whether this stationary point is a local maximum, a local minimum, or a point of inflection.

[5 marks]

8. Let z_1 and z_2 be the complex numbers given by $z_1 = 3+2j$ and $z_2 = -1+j$. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.

[6 marks]

9. State the value of $\cos^{-1}(\sqrt{3}/2)$ (you should give an exact answer in radians). Give the general solution of the equation

$$\cos\theta = \frac{\sqrt{3}}{2}.$$

[4 marks]

10. Let $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{k}$. Find $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a}\cdot\mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$ (in radians, to 3 decimal places)?

[6 marks]

SECTION B

11. Give the Maclaurin series expansion of the function $f(x) = e^x$ up to and including the term in x^4 (you are not required to show any working if you remember this expansion).

[2 marks]

Hence, or otherwise, determine the Maclaurin series expansions of the following functions, up to and including the terms in x^4:

$$\text{a) } xe^x; \qquad \text{b) } e^{2x}; \qquad \text{c) } (e^x)^2; \qquad \text{d) } e^{\left(x^2\right)}.$$

[10 marks]

Use the Maclaurin series expansion of e^x up to the term in x^4 to obtain an approximation to $e^{0.1}$. You should give your approximation to 6 decimal places.

[3 marks]

12. Calculate the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}\,3^n}\, x^n.$$

[8 marks]

Use the alternating series test to show that the series converges when $x = R$. Write down the series when $x = -R$, and determine whether it is convergent or divergent. Hence state all of the (real) values of x for which the power series converges.

[7 marks]

13. By sketching the graphs of $y = 2x^2$ and $y = \cos x$ on the same axes, explain why the equation

$$f(x) = 2x^2 - \cos x = 0$$

has exactly two solutions (for real x).

[8 marks]

Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = 0.6$ to obtain successive approximations x_1, x_2, and x_3 to a solution of the equation $f(x) = 0$: give each approximation to 6 decimal places.

Without doing any more calculations, give the best approximation you can to the second solution of the equation $f(x) = 0$.

[7 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 + x + 1 & \text{if } x \leq 0 \\ \frac{1}{1-x} & \text{if } x > 0. \end{cases}$$

Sketch the graph of $y = f(x)$, indicating clearly the positions of any zeros, stationary points, and asymptotes. (You will get no marks for sketching the graph unless you show how you have determined the positions of these features.)

[12 marks]

At which value or values of x is $f(x)$ not continuous? At which value or values is it not differentiable?

[3 marks]

15.

Let $z = \cos\theta + j\sin\theta$. By using de Moivre's theorem, show that

$$z^n + z^{-n} = 2\cos(n\theta)$$

for all positive integers n.

[4 marks]

By taking the fourth power of each side of this equation in the case $n = 1$, find the integers a, b, and c such that

$$8\cos^4\theta = a\cos(4\theta) + b\cos(2\theta) + c.$$

[5 marks]

Use this result to evaluate

$$\int_0^{\pi/6} \cos^4 x \, dx.$$

(You should either give an exact answer, or give your answer to 6 decimal places.)

[6 marks]