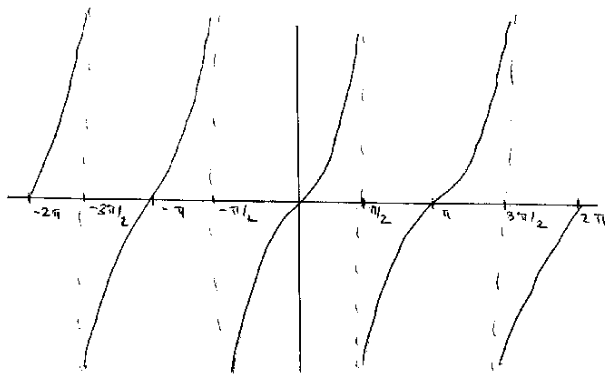
MATH191 Exam January 2002, Solutions

1. The range is $\mathbb{R}$.
The graph is shown below.



2. We have $f(0) = 2$, $f'(x) = \frac{1}{2}(4+x)^{-1/2}$, so $f'(0) = 1/4$, and $f''(x) = -\frac{1}{4}(4+x)^{-3/2}$, so $f''(0) = -1/32$.

Hence the first three terms in the Maclaurin series expansion of $f(x)$ are

$$f(x) = 2 + x/4 - x^2/64 + \cdots.$$

3.

a) This is a polynomial with both even and odd powers, so is neither even nor odd.

b) Let $f(x) = 1/(1+x^4)$. Then $f(-x) = 1/(1+(-x)^4) = 1/(1+x^4) = f(x)$. Hence $f(x)$ is even.

c) Let $f(x) = x\sin(x^2)$. Then $f(-x) = -x\sin((-x)^2) = -x\sin(x^2) = -f(x)$. Hence $f(x)$ is odd.

4.

$$\begin{aligned}\int_0^1 \left(e^{-x}+1\right)\,dx &= \left[-e^{-x}+x\right]_0^1 \\ &= e^0 - e^{-1} + 1 - 0 = 2 - 1/e = 1.632\end{aligned}$$

to three decimal places.

5.

a) $f(x) = (x^2+2x-1)/(x-2)$ is continuous at $x = 1$. Hence the limit exists and is equal to $f(1) = 2/(-1) = -2$.

b) By L'Hôpital's rule,

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin^2 x} = \lim_{x\to 0} \frac{-\sin x}{2\sin x \cos x} = \lim_{x\to 0} \frac{-1}{2\cos x} = -\frac{1}{2}.$$

Hence the limit exists and is equal to $-1/2$.

c) The limit does not exist.

6.

a) By the product rule,

$$\frac{d}{dx} x^2 \sinh x = 2x \sinh x + x^2 \cosh x.$$

b) By the chain rule,

$$\frac{d}{dx} \cos(1 - x^2) = -(-2x)\sin(1 - x^2) = 2x \sin(1 - x^2).$$

c) By the quotient rule,

$$\frac{d}{dx} \frac{e^x}{x} = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}.$$

7. $f'(x) = \frac{1}{x} - 2$. Stationary points are given by solutions of $f'(x) = 0$, or $\frac{1}{x} = 2$, or $x = 1/2$.

To determine its nature, $f''(x) = -1/x^2$, so $f''(1/2) < 0$, and the stationary point is a local maximum.

8.

$$\begin{aligned} z_1 + z_2 &= 3 + 2j. \\ z_1 - z_2 &= 1 + 4j. \\ z_1 z_2 &= 5 + j. \\ z_1/z_2 &= \frac{(2+3j)(1+j)}{(1-j)(1+j)} = \frac{-1+5j}{2}. \end{aligned}$$

9. $\cos^{-1}(1/2) = \pi/3$.

Hence the general solution of $\cos\theta = 1/2$ is

$$\theta = \pm\frac{\pi}{3} + 2n\pi \qquad (n \in \mathbb{Z}).$$

10.

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= 4\mathbf{i}-\mathbf{j}+2\mathbf{k}. \\
\mathbf{a}-\mathbf{b} &= -2\mathbf{i}+3\mathbf{j}. \\
|\mathbf{a}| &= \sqrt{1^2+1^2+1^2}=\sqrt{3}. \\
|\mathbf{b}| &= \sqrt{3^2+2^2+1^2}=\sqrt{14}. \\
\mathbf{a}\cdot\mathbf{b} &= 3-2+1=2.
\end{aligned}$$

Hence the angle between **a** and **b** is $\cos^{-1}(2/\sqrt{42}) = 1.257$ to 3 decimal places.

11. The Maclaurin series expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

Hence

a)

$$xe^x = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \cdots$$

b)

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \cdots$$

c) $(e^x)^2 = e^{2x}$, so the expansion is as in b).

d)

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \cdots$$

The approximation is $1 + \frac{1}{10} + \frac{1}{200} + \frac{1}{6000} + \frac{1}{240000} = 1.105171$ to 6 decimal places.

12. The radius of the convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case $a_n = 1/(n^2+1)$, so $|a_n/a_{n+1}| = ((n+1)^2 + 1)/(n^2+1) = (n^2+2n+2)/(n^2+1)$, which tends to 1 as $n \to \infty$. Hence $R = 1$.

When $x = -1$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}.$$

This converges by the alternating series test, which states that

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges if a_n is a decreasing sequence with $a_n \to 0$.

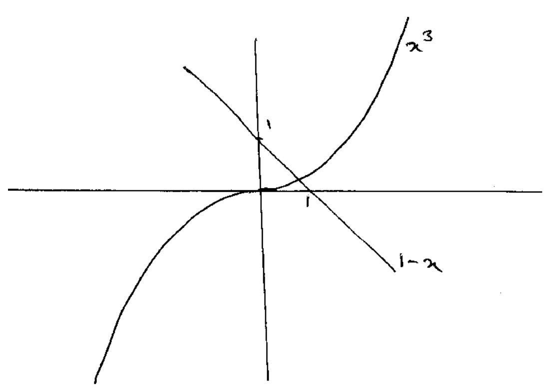
When $x = 1$, the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1},$$

which is convergent by comparison with $\sum \frac{1}{n^2}$ (whose convergence is a standard result).

Hence the series converges if and only if $-1 \le x \le 1$.

13. The graphs are as shown:



Since x^3 is increasing and $1-x$ is decreasing, there can be at most one solution to $x^3 = 1-x$. Since the functions take values 0 and 1 at $x = 0$, and values 1 and 0 at $x = 1$, there is a solution in $[0,1]$.

Setting $f(x) = x^3 + x - 1$, we have $f'(x) = 3x^2 + 1$, so the Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}.$$

Hence

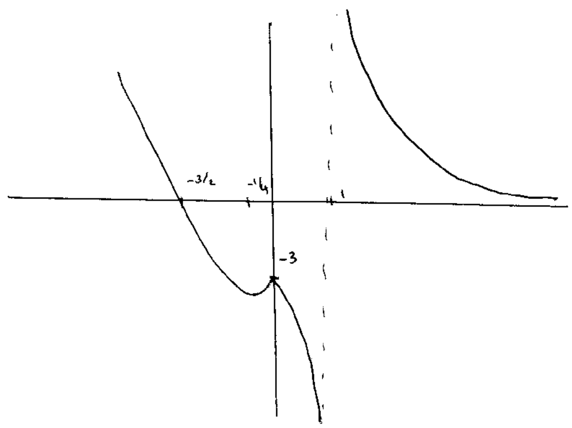
$$x_1 = 0.688461, \quad x_2 = 0.682359, \quad \text{and } x_3 = 0.682327$$

to 6 decimal places.

14. For $x < 0$ we have $f(x) = 2x^2 + x - 3 = (x-1)(2x+3)$, which has a zero at $x = -3/2$. The derivative is $f'(x) = 4x + 1$, so there is a stationary point at $x = -1/4$. Since $f''(x) = 4$, the stationary point is a local minimum. $f(x) = -3\frac{1}{8}$ at the stationary point. The gradient of $2x^2 + x - 3$ at $x = 0$ is 1.

For $x \geq 0$ we have $f(x) = 3/(x-1)$, which has no zeros and is equal to -3 at $x = 0$, and tends to 0 as $x \to \infty$. $f'(x) = -3/(x-1)^2$, so there are no stationary points, and $f(x)$ is decreasing in $(0,1) \cup (1,\infty)$; the gradient is -3 at $x = 0$. There is a vertical asymptote at $x = 1$.

The graph of $f(x)$ is therefore



$f(x)$ is not continuous at $x = 1$, since 1 is not in its maximal domain.

$f(x)$ is not differentiable at $x = 1$ (not in maximal domain), and at $x = 0$ (no well-defined tangent to the graph at this point).

15. By de Moivre's theorem and the binomial theorem

$$\cos 4\theta + i \sin 4\theta = (c + is)^4 = c^4 + 4ic^3 s - 6c^2 s^2 - 4ics^3 + s^4,$$

where $c = \cos\theta$ and $s = \sin\theta$.

Equating real parts gives

$$\cos 4\theta = c^4 - 6c^2 s^2 + s^4.$$

By Pythagoras's theorem, $s^2 = (1 - c^2)$, so

$$\cos 4\theta = c^4 - 6c^2(1 - c^2) + (1 - c^2)^2 = 8c^4 - 8c^2 + 1.$$

Hence $a = 8$, $b = -8$, and $c = 1$.

Equating imaginary parts gives

$$\sin 4\theta = 4c^3 s - 4cs^3 = 4 \tan\theta (c^4 - c^2 s^2).$$

Using $c^2 = 1 - s^2$, this gives

$$\sin 4\theta = 4 \tan\theta ((1 - s^2)^2 - (1 - s^2)s^2) = 4 \tan\theta (2s^4 - 3s^2 + 1).$$

Hence $d = 2$, $e = -3$, and $f = 1$.

When $\theta = \pi/4$, $\sin\theta = \cos\theta = 1/\sqrt{2}$, so $s^2 = c^2 = 1/2$.

Thus the RHS of the identity for $\cos 4\theta$ becomes $8/2^2 - 8/2 + 1 = 2 - 4 + 1 = -1$, which checks since the LHS is $\cos\pi = -1$.

Since $\tan\pi/4 = 1$, the RHS of the identity for $\sin 4\theta$ becomes $4(2/4 - 3/2 + 1) = 4(1/2 - 3/2 + 1) = 0$, which checks since the LHS is $\sin\pi = 0$.