SECTION A

1. State the range of the function

$$f(x) = \tan x.$$

Sketch the graph of $y = f(x)$ for $-2\pi \leq x \leq 2\pi$, indicating the values of x where it crosses the x-axis, and where there are vertical asymptotes.

[4 marks]

2. By evaluating $f(0)$, $f'(0)$, and $f''(0)$, obtain the Maclaurin series expansion of the function

$$f(x) = \sqrt{4 + x}$$

up to and including the term in x^2.

[5 marks]

3. State, with reasons, whether the following functions are odd, even, or neither:

$$\text{a) } 3x^2 + 2x - 1; \qquad \text{b) } \frac{1}{1 + x^4}; \qquad \text{c) } x \sin(x^2).$$

[6 marks]

4. Calculate the integral

$$\int_0^1 \left(e^{-x} + 1\right)\, \mathrm{d}x,$$

evaluating the result to three decimal places.

[5 marks]

5. Which of the following limits exist? Evaluate those which do.

$$\text{a) } \lim_{x \to 1} \frac{x^2 + 2x - 1}{x - 2}; \qquad \text{b) } \lim_{x \to 0} \frac{\cos(x) - 1}{\sin^2 x}; \qquad \text{c) } \lim_{x \to \infty} \cos(2x).$$

[8 marks]

6. Differentiate the following functions:

$$\text{a) } x^2 \sinh x; \qquad \text{b) } \cos(1 - x^2); \qquad \text{c) } \frac{e^x}{x}.$$

[6 marks]

7. Show that the function

$$f(x) = \ln(x) - 2x + 1 \qquad (x > 0)$$

has exactly one stationary point. Determine whether this stationary point is a local maximum, a local minimum, or a point of inflection.

[5 marks]

8. Let z_1 and z_2 be the complex numbers given by $z_1 = 2 + 3j$ and $z_2 = 1 - j$. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.

[6 marks]

9. State the value of $\cos^{-1}(1/2)$ (you should give an exact answer in radians). Give the general solution of the equation

$$\cos\theta = \frac{1}{2}.$$

[4 marks]

10. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$ (in radians, to 3 decimal places)?

[6 marks]

SECTION B

11. Give the Maclaurin series expansion of the function $f(x) = e^x$ up to and including the term in x^4 (you are not required to show any working if you remember this expansion).

[2 marks]

Hence, or otherwise, determine the Maclaurin series expansions of the following functions, up to and including the terms in x^4:

a) xe^x; b) e^{2x}; c) $(e^x)^2$; d) $e^{(x^2)}$.

[10 marks]

Use the Maclaurin series expansion of e^x up to the term in x^4 to obtain an approximation to $e^{0.1}$. You should give your approximation to 6 decimal places.

[3 marks]

12. Calculate the radius of convergence R of the power series

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1}\, x^n.$$

[8 marks]

Write down the series when $x = R$ and when $x = -R$, and state in each case whether it is convergent or divergent, giving brief reasons for your answers. Hence state all of the (real) values of x for which the power series converges.

[7 marks]

13. By sketching the graphs of $y = x^3$ and $y = 1 - x$ on the same axes, explain why the equation

$$f(x) = x^3 + x - 1 = 0$$

has exactly one solution. Explain why this solution must lie in $[0, 1]$.

[7 marks]

Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = 0.6$ to obtain successive approximations x_1, x_2, and x_3 to this solution. You should give each approximation to 6 decimal places.

[8 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} \frac{3}{x-1} & \text{if } x \geq 0 \\ 2x^2 + x - 3 & \text{if } x < 0. \end{cases}$$

Sketch the graph of $y = f(x)$, indicating clearly the positions of any zeros, stationary points, and asymptotes. (You will get no marks for sketching the graph unless you show how you have determined the positions of these features.)

[12 marks]

For which value or values of x is $f(x)$ not continuous? For which value or values is it not differentiable?

[3 marks]

15.

By using de Moivre's theorem, find the integers a, b, c, d, e, and f such that

$$\cos 4\theta = a\cos^4\theta + b\cos^2\theta + c$$

and

$$\sin 4\theta = 4\tan\theta(d\sin^4\theta + e\sin^2\theta + f).$$

[11 marks]

Check both of these results when $\theta = \frac{\pi}{4}$ (you should work with exact values of $\cos\theta$ and $\sin\theta$, rather than evaluating them on your calculator).

[4 marks]