

SECTION A

1. State the maximal domain of the function

$$f(x) = \ln(x).$$

Sketch the graph of $y = f(x)$.

[4 marks]

2. Obtain the Maclaurin series expansion of the function

$$f(x) = \frac{1}{\sqrt{1+x}}$$

up to and including the term in x^2 .

[5 marks]

3. State, with reasons, whether the following functions are odd, even, or neither:

a) $\sin(1+x^2)$; b) $\frac{1-x}{1+x^4}$; c) $\sinh x \cos x$.

[6 marks]

4. Calculate the integral

$$\int_0^{\pi/2} (\cos x + 2e^x) dx,$$

evaluating the result to three decimal places.

[5 marks]

5. Which of the following limits exist? Evaluate those which do.

a) $\lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{x - 2}$; b) $\lim_{x \rightarrow 0} \frac{\sin^2 x}{1 - \cos x}$; c) $\lim_{x \rightarrow \infty} \sin(2x)$.

[8 marks]

6. Differentiate the following functions:

a) $e^x \cos x$; b) $\tan(1 + x^2)$; c) $\frac{\ln x}{\cos x}$.

[6 marks]

7. Show that the function

$$f(x) = \ln x - 2x^2 \quad (x > 0)$$

has one (and only one) stationary point. Determine whether this stationary point is a local maximum, a local minimum, or a point of inflection.

[5 marks]

8. Consider the curve defined by

$$2x^2y - xy^2 + 3x - 2y + 2 = 0.$$

Find an expression for $\frac{dy}{dx}$, and hence give the equation of the tangent to the curve at the point $(x, y) = (0, 1)$.

[6 marks]

9. Find the general solution (for real θ) of the equation

$$3 \cos \theta = 2.$$

Give your answer in radians to three decimal places.

[4 marks]

10. Let $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. Calculate the angle between \mathbf{a} and \mathbf{b} in radians to three decimal places.

[6 marks]

SECTION B

11. Calculate the Maclaurin series expansion of the function $f(x) = \ln(1+x)$ up to the term in x^3 . Use these terms to obtain an approximation to $\ln(1.1)$.

[8 marks]

Calculate the remainder term

$$\frac{f^{(4)}(c)}{4!}x^4$$

for this approximation, and state the range of values that c must lie in. Use this to show that your approximation to $\ln(1.1)$ is accurate to at least 4 decimal places.

[7 marks]

12. Calculate the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n 3^n} x^n.$$

[8 marks]

Use the alternating series test to show that the series converges when $x = R$. Write down the series when $x = -R$, and state whether it is convergent or divergent. Hence state all of the (real) values of x for which the power series converges.

[7 marks]

13. By sketching the graphs of $y = x^2$ and $y = -\ln x$ on the same axes, explain why the equation

$$f(x) = x^2 + \ln x = 0$$

has exactly one solution (for real x).

[7 marks]

Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = 0.5$ to obtain successive approximations x_1 , x_2 , and x_3 to this solution. You should give each approximation to 6 decimal places.

[8 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 - 4x & \text{if } x \geq 0 \\ \frac{x}{x+1} & \text{if } x < 0. \end{cases}$$

Sketch the graph of $y = f(x)$, indicating clearly the positions of any zeros, stationary points, and asymptotes. (You will get no marks for sketching the graph unless you show how you have determined the positions of these features.)

[12 marks]

For which value or values of x is $f(x)$ not continuous? For which value or values is it not differentiable?

[3 marks]

15.

By using de Moivre's theorem, find the integers a , b , c , d , e , and f such that

$$\cos 4\theta = a \cos^4 \theta + b \cos^2 \theta + c$$

and

$$\sin 4\theta = 4 \tan \theta (d \sin^4 \theta + e \sin^2 \theta + f).$$

[11 marks]

Check both of these results when $\theta = \frac{\pi}{4}$ (you should work with exact values of $\cos \theta$ and $\sin \theta$, rather than evaluating them on your calculator).

[4 marks]