PAPER CODE NO.
MATH186

THE UNIVERSITY *of* LIVERPOOL

SUMMER 2006 EXAMINATIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Chemistry : Year 1
Master of Physics : Year 1

MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55 % of the available marks. The vectors $\mathbf{i}$, $\mathbf{j}, \mathbf{k}$ are the unit vectors in the directions of the coordinate axes Ox, Oy, and Oz respectively.

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SECTION A

1. For the matrices $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 4 & 1 & 6 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 8 \\ 2 & 1 \\ 1 & 1 \\ 12 & 6 \end{bmatrix},$$

evaluate $\mathbf{AB}$ and $\mathbf{BA}$. Comment on your result. [6 marks]

2. Find the magnitude of the vector $2\mathbf{a} + \mathbf{b} - \mathbf{c}$, where

$$\mathbf{a} = -4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}, \quad \mathbf{c} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

[5 marks]

3. Find the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$, where

$$\mathbf{a} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

Hence, find the angle in degrees between these vectors. Give your answer to three decimal places. [5 marks]

4. The position vector of a particle at time t is $\mathbf{r}(t)$, where

$$\mathbf{r}(t) = t\sin(t)\mathbf{i} + t\cos(t)\mathbf{j} + \mathbf{k}.$$

Find the particle's velocity, speed and acceleration at time t. [4 marks]

5. A force $\mathbf{F}$ of magnitude 5 N acting in the direction of $3\mathbf{i} - 4\mathbf{k}$ moves a particle along a straight line from the point with coordinates $(1, 2, -1)$ m to the point with coordinates $(7, -2, 2)$ m. Find the work done by this force. [5 marks]

6. For the vectors $\mathbf{a}$ and $\mathbf{b}$, where

$$\mathbf{a} = -3\mathbf{i} + 6\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k},$$

verify that

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b}.$$

[6 marks]

7. For the scalar field φ, where

$$\varphi(x, y, z) = x^3 y^2 e^z,$$

find grad φ and verify that curl(grad φ) $= \mathbf{0}$. [4 marks]

8. Find the general solution of each of the following differential equations:

(i) $\sin(2x)\dfrac{dy}{dx} + 2\sin^2(x)y = 2\sin(x)$ [5 marks]

(ii) $(x - 2y)\dfrac{dy}{dx} = 2x - y$ [8 marks]

9. Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$$

given that $y = 1$ and $dy/dx = 0$ when $x = 0$. What is the value of y when $x = \pi/\sqrt{3}$? Give your answer to three decimal places. [7 marks]

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SECTION B

10. (a) For the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{bmatrix} -3 & 1 & 0 \\ 2 & 0 & 0 \\ 5 & -1 & 4 \end{bmatrix},$$

find $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$. Verify that $\det(\mathbf{A}^{-1}) = 1/(\det(\mathbf{A}))$.

[6 marks]

(b) Use Cramer's Rule to solve the system of equations

$$\begin{aligned} 2x - y + 3z &= 4 \\ x + 9y - 2z &= -8 \\ 4x - 8y + 11z &= 15 \end{aligned}$$

Check your answer by direct substitution. [9 marks]

11. (a) Points $A,\ B,\ C$ have position vectors

$$\overrightarrow{OA} = (1, -3, 7),\ \overrightarrow{OB} = (2, 1, 1),\ \overrightarrow{OC} = (6, -1, 2).$$

(i) Find the area of the triangle ABC. [5 marks]

(ii) Find two unit vectors perpendicular to the plane of that triangle. [2 marks]

(iii) Find the length of the altitude from the vertex A to the side BC. [3 marks]

(b) If $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$, $\mathbf{c} = -2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, find the volume of the parallelepiped with sides $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. [5 marks]

12. A particle of mass 2 kg is moving under the action of the force

$$\mathbf{F} = -4\cos(t)\mathbf{i} - 4\sin(t)\mathbf{j}\ \text{N}.$$

Find the position $\mathbf{r}(t)$ (in metres) of the particle at time t (in seconds) given that at time $t = 0$ s the particle is located at the point $(2, 0, 0)$ and has a velocity of $2\mathbf{j} + \mathbf{k}$ ms^{-1}. Calculate the displacement and distance travelled by the particle over the time interval $0 \leq t \leq 2\pi$.

[15 marks]

13. (a) For the vector fields $\mathbf{F}$ and $\mathbf{G}$, where

$$\mathbf{F} = 2xy\mathbf{i} + e^y\mathbf{j} + 2z\mathbf{k}, \quad \mathbf{G} = -2e^z\mathbf{i} - y^2z\mathbf{j} + 2\mathbf{k},$$

verify the following identity

$$\mathrm{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\mathrm{curl}\mathbf{F}) - \mathbf{F} \cdot (\mathrm{curl}\mathbf{G}).$$

[7 marks]

(b) Given that $\varphi(x, y, z) = x^2 + y^2 + z^2$, and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector, show that $\mathrm{curl}(\varphi\mathbf{r}) = \mathbf{0}$. [4 marks]

(c) Given that $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a constant vector, and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector, find $\mathbf{r} \cdot \mathbf{a}$ and $\mathbf{r} - \mathbf{a}$. Hence, show that

$$\mathrm{grad}(\mathbf{r} \cdot \mathbf{a}) = \mathbf{a}, \quad \mathrm{div}(\mathbf{r} - \mathbf{a}) = 3, \quad \mathrm{curl}(\mathbf{r} - \mathbf{a}) = \mathbf{0}.$$

[4 marks]

14. (a) The displacement $x(t)$ of a particle satisfies the differential equation

$$\frac{d^2x}{dt^2} - 4x = 8t^2 - 2t.$$

The initial displacement is 0 m and the initial speed is 0 ms^{-1}. Find the displacement at time t. [10 marks]

(b) Show that the polynomial $P(x)$, where

$$P(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

satisfies the Legendre differential equation

$$(1 - x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + 20P = 0.$$

Verify that

$$\int_{-1}^{1} P^2(x)dx = \frac{2}{9}.$$

[5 marks]