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SECTION A

1. Find all solutions (if there are any) of the systems of equations

$$\text{(i)}\quad \begin{aligned} x-2y+z &= 2\\ 2x-4y+z &= 3\\ -3x+6y-z &= 1 \end{aligned} \qquad \text{(ii)}\quad \begin{aligned} x-2y+z &= 2\\ 2x-4y+z &= 3\\ -3x+6y-z &= -4 \end{aligned}$$

[11 marks]

2. (a) Find the greatest common divisor d of 507 and 1677, and find integers s and t such that

$$d = 507s + 1677t.$$

(b) Find the inverse of 35 mod 79.

[11 marks]

3. Find all solutions (if any) of each of the following congruences

(i) $10x \equiv 15 \bmod 43$.

(ii) $10x \equiv 15 \bmod 44$.

(iii) $10x \equiv 15 \bmod 45$.

[11 marks]

4. Let π and ρ be the permutations

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} \text{ and } \rho = (261)(34).$$

Write $\pi^2, \pi\rho, \pi^{-1}\rho$ and ρ^3 as products of disjoint cycles and find the order and sign of each of these permutations. [11 marks]

5. Find the smallest positive number x which satisfies the two simultaneous congruences

$$x \equiv 5 \bmod 27 \quad \text{and} \quad x \equiv 4 \bmod 11.$$

Find also the next smallest x which satisfies both congruences. [11 marks]

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SECTION B

6. Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and show that 2 is an eigenvalue of A.

Find all the eigenvalues and eigenvectors of A. [15 marks]

7. (i) Outline the method of coding and decoding messages using a public key code with base n and coding exponent a.

(ii) A code with base $221 = 13 \times 17$ and exponent 55 is used to encode a message, with letter - digit correspondence

T	H	E	L	A	Z	Y	D	O	G
0	1	2	3	4	5	6	7	8	9

Find the decoding exponent.

Hence decode the message 219/ 174 .

[15 marks]

8. (i) Let G be a group. Say what it means for G to be *cyclic*.

(ii) Show that the group G_{14} of invertible congruence classes mod 14 is cyclic.

(iii) Write down the number of elements of each group in the following list, where C_n denotes a cyclic group of order n, G_n denotes the group of invertible congruence classes under multiplication mod n and $S(n)$ denotes the group of permutations of n numbers.

$$C_6,\ C_2 \times C_2,\ G_{14},\ G_5,\ S(3).$$

Decide, giving reasons, which pairs of groups in the list are isomorphic.

[15 marks]

9. List the code words of the group code with generator matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

State how many errors are detected and how many are corrected by this code, giving reasons for your answers. Give a table of syndromes for this code for all possible single digit errors in transmission.

Suppose that eight letters are represented by binary numbers as shown in the table below.

O	*D*	*E*	*S*	*P*	*A*	*I*	*R*
000	001	010	011	100	101	110	111

An encoded message is received as

1011101 1011000 1011011 1000110 0110101 0101011 0111110.

Find the original message. [15 marks]