PAPER CODE NO.
MATH181

THE UNIVERSITY
of LIVERPOOL

January 2007 EXAMINATIONS

Bachelor of Science : Year 1
Master of Chemistry : Year 1
Master of Earth Sciences : Year 1
Master of Physics : Year 1

METHODS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
Section A carries 55% of the available marks.

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SECTION A

1. A function is defined by

$$f(x) = e^{-|x|/4}$$

for $-4 \le x \le 4$. Sketch this function.

[**3 marks**]

2. What is a one-to-one function? A function is given by

$$y = f(x) = \frac{x+3}{x-5}.$$

What is the domain and range? Find $f^{-1}(x)$ given that the function $f(x)$ is one-to-one.

[**5 marks**]

3. Differentiate the following with respect to x

$$\text{(i)} \quad 3x^8 e^{-5x} \quad , \quad \text{(ii)} \quad \frac{x^2}{(4x+5)^2} \quad , \quad \text{(iii)} \quad \sin(2x)\cosh^3(4x) \ .$$

[**9 marks**]

4. Suppose that two variables satisfy the equation

$$x^3 - y^3 x^2 + \frac{5}{x}\cos(y) = 7.$$

Find implicitly $\frac{dy}{dx}$ in terms of y and x.

[**5 marks**]

5. Determine the following indefinite integrals

$$\text{(i)} \quad \int (x^2 + 3x^4) dx \quad , \quad \text{(ii)} \quad \int \cosh^2(3x) dx \ .$$

[**5 marks**]

6. Evaluate

$$\text{(i)} \quad \int_0^{\pi} \cos^2 x dx \quad , \quad \text{(iii)} \quad \int_0^{\infty} x e^{-3x^2} dx \ .$$

[**5 marks**]

7. (i) Evaluate the sum

$$\sum_{k=0}^{5}(3(2)^k + 5k).$$

(ii) What is the limit of the sequence

$$S_n = \frac{3n^4 + 12n^2 + 12}{n^4 - 4}$$

as $n \to \infty$.

[**6 marks**]

8. State de Moivre's theorem. If $z_1 = 2 + 3i$ and $z_2 = 1 + 2i$ evaluate $|z_1 z_2|$. Write $\ln(-1)$ in terms of complex numbers.

[**5 marks**]

9. Suppose that

$$d(x, y, z, t) = t^3(x^2 + y^2 - z^2 - t^2).$$

What are $\frac{\partial d}{\partial x}$, $\frac{\partial d}{\partial t}$, $\frac{\partial d}{\partial y}$, and $\frac{\partial^2 d}{\partial x^2}$?

[**4 marks**]

10. Consider the function $\cos^2(3x) - \sin^2(3x)$. Obtain the Maclaurin series expansion of this function up to and including the term x^4.

[**4 marks**]

11. Five particles decay electromagnetically with half lives of (30, 40, 55, 60, and 80)$\times 10^{-19} s$. What is the mean half life and the variance of the half lives?

[**4 marks**]

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SECTION B

12.

(i) Evaluate the definite integral

$$\int_0^1 \frac{2x^2+5x}{(x+2)(x+3)}dx.$$

[6 marks]

(ii) Evaluate the integral

$$\int x^2\sqrt{dx^2+dy^2}$$

on the curve $x=\cos\theta$ and $y=\sin\theta$ between $\theta=0$ and $\theta=2\pi$.

[3 marks]

(iii) Using a suitable substitution or otherwise evaluate the indefinite integral

$$\int \frac{(\cos^2 x-\sin^2 x)}{(1-4\sin 2x)^2}dx.$$

[6 marks]

13. (i) By using polar coordinates or otherwise, integrate the function

$$F(x,y)=2+(x^2+y^2)^2$$

over the area enclosed by the curve $x^2+y^2=1$.

[6 marks]

(ii) Evaluate the integral

$$\int_A \left(xy+x^2y^2+3\right)dxdy$$

where the area A is bounded by the lines $y=0$, $x=0$ and $y=1-x$.

[9 marks]

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14. (i) Find in polar form all the roots of the equation

$$z^4 = 16(1+i)$$

and draw a diagram showing their position in the complex plane.

[**6 marks**]

(ii) Use De Moivre's theorem to express $\cos^5\theta$ in terms of $\cos 5\theta$, $\cos 3\theta$, and $\cos\theta$.

[**6 marks**]

(iii) Using this result determine

$$\int \cos^5\theta d\theta.$$

[**3 marks**]

15. (i) Suppose that

$$I_n = \int_0^\infty x^n e^{-ax^2} dx$$

where n is a positive integer. What is the value of I_1? By differentiating with respect to a determine I_{n+2} in terms of I_n. Hence determine I_3.

[**6 marks**]

(ii) Using the Maclaurin series expansion to the first three terms of $\cosh(x^2)$, compute the approximate value of

$$\int_0^1 \cosh(x^2)dx.$$

[**5 marks**]

(iii) Show that

$$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}.$$

by using implicit differentiation.

[**4 marks**]

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16. (i) How many distinct arrangements are there of the word "MATHEMATICS"?

[**3 marks**]

(ii) Find the number of ways a magician can choose eight of the twelve white rabbits that are sitting in his top hat.

[**3 marks**]

(iii) A gas consists of two types of particles each with a Maxwellian distribution that are totally independent. It is enclosed in a box with a hole, such that the gas escapes slowly. For particle A the probability distribution is.

$$P(v)_A = 2\sqrt{\frac{m}{2\pi}}e^{-mv^2/2},\ 1 = \int_0^\infty dv P(v)_A,$$

and for particle B

$$P(v)_B = 2\sqrt{\frac{M}{2\pi}}e^{-Mv^2/2},\ 1 = \int_0^\infty dv P(v)_B.$$

What is the average speed of a particle in the box? Sketch the two probability distributions on the same graph assuming $M > m$. What is the speed, $v_{coincide}$, for which the probability distributions coincide? If the escaping particles speed satisfies $v \geq v_{coincide}$, calculate the average speed of particle A escaping from the box.

[**9 marks**]