PAPER CODE NO.
MATH176

THE UNIVERSITY *of* LIVERPOOL

SUMMER 2006 EXAMINATIONS

Bachelor of Science	:	Year 1
Bachelor of Science	:	Year 2
Master of Chemistry	:	Year 1
Master of Physics	:	Year 1

VECTORS AND DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Candidates should answer FOUR questions out of SIX. The vectors **i**, **j**, **k** are the unit vectors in the directions of the coordinate axes Ox, Oy and Oz, respectively.

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1. (a) For the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, where

$$\mathbf{a} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{c} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k},$$

find the magnitude of the vector $2\mathbf{a} + \mathbf{b} - \mathbf{c}$. [5 marks]

Find the projection of the vector $\mathbf{c}$ onto the line passing through the points $(-6, 1, 1)$ and $(3, 2, -4)$. [5 marks]

Find the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$. Hence, find the angle in degrees between these vectors. Give your answer to three decimal places. [5 marks]

Show that the vector

$$(|\mathbf{b}|\mathbf{a} + |\mathbf{a}|\mathbf{b})/(|\mathbf{a}||\mathbf{b}|)$$

bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

[**Hint:** Show that this vector makes the same angle with $\mathbf{a}$ as it does with $\mathbf{b}$.] [5 marks]

(b) The points $A(1, -2, 1)$, $B(0, 1, 6)$ and $C(-3, 4, -2)$ form a triangle. The line from the vertex A to the midpoint M of the side BC is called a *median*. Show that $\overrightarrow{AM} = (\overrightarrow{AB} + \overrightarrow{AC})/2$. Hence, find the distance between the vertex A and the point M. [5 marks]

2. (a) For the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, where

$$\mathbf{a} = -3\mathbf{i} + 6\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \mathbf{c} = -4\mathbf{i} + 6\mathbf{k}, \quad \mathbf{d} = \mathbf{i} - 2\mathbf{j} + 7\mathbf{k},$$

verify that

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b},$$

and

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

[13 marks]

(b) Find the area of the triangle with vertices $A(1, -3, 7)$, $B(2, 1, 1)$ and $C(6, -1, 2)$. Find two unit vectors perpendicular to the plane of that triangle. [7 marks]

(c) If $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$, $\mathbf{c} = -2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, find the volume of the parallelepiped with sides $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. [5 marks]

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3. (a) Find the work done by the force $\mathbf{F} = 3\mathbf{i} - 4\mathbf{k}$ N in moving an object along a straight line from the point with coordinates $(1, 2, -1)$ m to the point with coordinates $(7, -2, 2)$ m. [3 marks]

Given that $\mathbf{F}$ is applied at the point $(2, 0, 1)$ m, find the moment vector $\mathbf{m}$ of $\mathbf{F}$ about the origin. [3 marks]

(b) The position vector of a particle at time t is $\mathbf{r}(t)$, where

$$\mathbf{r}(t) = t \sin(t)\mathbf{i} + t \cos(t)\mathbf{j} + \mathbf{k}.$$

Find the particle's velocity, speed and acceleration at time t. [4 marks]

(c) A particle of mass 2 kg is moving under the action of the force

$$\mathbf{F} = -4\cos(t)\mathbf{i} - 4\sin(t)\mathbf{j} \text{ N}.$$

Find the position $\mathbf{r}(t)$ (in metres) of the particle at time t (in seconds) given that at time $t = 0$ s the particle is located at the point $(2, 0, 0)$ and has a velocity of $2\mathbf{j} + \mathbf{k}$ ms^{-1}.

[9 marks]

Calculate the displacement and distance travelled by the particle over the time interval $0 \le t \le 2\pi$. [6 marks]

4. (a) Find the general solution of each of the following differential equations:

(i) $2y\dfrac{dy}{dx} = e^{x-y^2}$ [4 marks]

(ii) $\sin(2x)\dfrac{dy}{dx} + 2\sin^2(x)y = 2\sin(x)$ [5 marks]

(iii) $(x-2y)\dfrac{dy}{dx} = 2x - y$ [8 marks]

(b) The differential equation governing the velocity v of a falling object of mass m subjected to air resistance proportional to the instantaneous velocity is given by

$$m\frac{dv}{dt} = mg - kv,$$

where g is the acceleration due to gravity, and k is a positive constant. If $m = 50$ kg, $g = 9.8$ m/sec^2, $k = 2$ kg/sec, and $v(0) = 3$ m/sec, find the velocity v at time t. [6 marks]

Find the acceleration at $t = 1$ sec. [2 marks]

5. (a) For the scalar field φ, where

$$\varphi(x, y, z) = x^3 y^2 e^z,$$

find grad φ and verify that curl(grad φ) $= \mathbf{0}$. [4 marks]

(b) For the vector fields $\mathbf{F}$ and $\mathbf{G}$, where

$$\mathbf{F} = 2xy\mathbf{i} + e^y\mathbf{j} + 2z\mathbf{k}, \quad \mathbf{G} = -2e^z\mathbf{i} - y^2z\mathbf{j} + 2\mathbf{k},$$

verify the following identity

$$\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\text{curl } \mathbf{F}) - \mathbf{F} \cdot (\text{curl } \mathbf{G}).$$

[7 marks]

(c) Given that $\varphi(x, y, z) = x^2 + y^2 + z^2$, and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector, show that curl($\varphi\mathbf{r}$) $= \mathbf{0}$. [4 marks]

(d) Given that $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a constant vector, and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector, find $\mathbf{r} \cdot \mathbf{a}$ and $\mathbf{r} - \mathbf{a}$. Hence, show that

$$\text{grad}(\mathbf{r} \cdot \mathbf{a}) = \mathbf{a}, \quad \text{div}(\mathbf{r} - \mathbf{a}) = 3, \quad \text{curl}(\mathbf{r} - \mathbf{a}) = \mathbf{0}.$$

[4 marks]

(e) For the scalar field φ and the vector field $\mathbf{F}$, where

$$\varphi(x, y, z) = \sin(xz), \quad \mathbf{F} = -yz\mathbf{j} - 6x^3\mathbf{k},$$

find div($\varphi\mathbf{F}$) and curl($\varphi\mathbf{F}$). [6 marks]

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6. (a) Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$$

given that $y = 1$ and $dy/dx = 0$ when $x = 0$. What is the value of y when $x = \pi/\sqrt{3}$? Give your answer to three decimal places. [7 marks]

(b) The displacement $x(t)$ of a particle satisfies the differential equation

$$\frac{d^2x}{dt^2} - 4x = 8t^2 - 2t.$$

Find the displacement $x(t)$ (in metres) of the particle at time t (in seconds) given that at time $t = 0$ s the displacement of the particle is 0 m and its speed is 0 ms^{-1}. [10 marks]

(c) Show that the polynomial $P(x)$, where

$$P(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

satisfies the Legendre differential equation

$$(1 - x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + 20P = 0.$$

Verify that

$$P(x) = \frac{1}{384}\frac{d^4}{dx^4}\left((x^2 - 1)^4\right),$$

and

$$\int_{-1}^{1} P^2(x)dx = \frac{2}{9}.$$

[8 marks]