THE UNIVERSITY
of LIVERPOOL

INSTRUCTIONS TO CANDIDATES

Answer *all* questions in Section A

The marks for the *best three* answers in Section B will be used in the assessment

Normal and Chi-squared tables are provided at the end of the paper

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Some Useful Formulae

1) For any two events A and B

$P(A \cup B) = P(A) + P(B) - P(A \cap B),$

$P(A \cap B) = P(A \mid B)P(B),$

$P(A \cap \overline{B}) = P(A) - P(A \cap B).$

2) If X has a Binomial distribution with parameters n and p

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad (x = 0,1,\ldots,n)$$

where

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

and for each integer $x \geq 1, x! = x(x-1)(x-2)\ldots 1, 0! = 1$.

Also, E(X)=np, V(X)=np(1-p).

Moreover, under suitable conditions,

$$P(a \leq X \leq b) = \Phi(\beta) - \Phi(\alpha),$$

where $\beta = \dfrac{(b + 0.5 - np)}{\{np(1-p\}^{1/2}}, \qquad \alpha = \dfrac{a - 0.5 - np}{\{np(1-p\}^{1/2}},$

and $\Phi(z)$ denotes the area to the left of z for a standard Normal distribution.

3) If X has a Poisson distribution with mean λ

$$P(X = x) = \frac{\lambda^x}{x!}\exp(-\lambda) \qquad (x = 0,1,\ldots),$$

and

$$E(X) = \lambda, V(X) = \lambda.$$

Section A

1. Marks, out of hundred, obtained in two subjects, I and II by a group of students is listed below, where, for convenience, the marks have been arranged in increasing order:

I:	35	41	48	51
	56	58	60	68
	69	73		

II:	19	25	28	36
	47	59	66	69
	72	92		

a) Obtain a back-back stem and leaf plot of the data [4 marks]

b) Calculate the median marks for the two subjects [2 marks]

c) In the light of your results in a) and b) above, comment on the relative performance of this particular group of students on subjects I and II. [2 marks]

2. A student newspaper publishes three feature columns, called "Environment" (E), "Cinema" (C), and "Sports" (S).
The regular reading habits of a randomly selected student with respect to these three columns are as follows:

Regular reader	E	C	S	$E \cap S$	$E \cap C$	$S \cap C$	$E \cap C \cap S$
Probability	0.14	0.23	0.37	0.09	0.08	0.13	0.05

At a council meeting of the student body running the newspaper, it was decided to review the readership of these three columns.

a) For a randomly selected student, find the probability that the student reads

i) at least one of the two 'leisure' columns on Cinema and Sports; [2 marks]

ii) at least one of the three columns. [4 marks]

b) A student is known to read regularly the Cinema column. Determine the conditional probability that this student also reads the Environment column.

[2 marks]

3. A news vendor orders 4 copies of a certain magazine per week. The number of individuals, X, who come in to buy this magazine in a week follows a Poisson distribution with mean 4.

Find the probability distribution of the random variable, Y, where Y is the number of copies of the magazine that are actually sold in a week. [6 marks]

Find the expected value, E(Y), of Y. [2 marks]

4. The employees of a building management firm are being offered a profit sharing /salary scheme.

Suppose that an individual currently receives an annual salary of £B. Under the scheme, this salary would be replaced by a basic salary of £(0.8)B plus a contribution £X_1 related to the firm's profitability over the next 12 months. Current forecasts predict that £X_1 will be £(0.4)B. The uncertainties of future operations imply however that £X_1 should be regarded as a random variable, normally distributed, with expected value £(0.4)B and standard deviation £(0.2)B.

Evaluate the probability that over a single year, an employee accepting the new scheme would have total earnings greater than the present earnings. [6 marks]

Suppose that an employee, Jack, say, takes a bet with a colleague, Charles, say according to which Jack would pay Charles £100 if his total earnings after 12 months are less than his current earnings; conversely, Charles would pay Jack £50 if Jack's total earnings exceed his current earnings.

Evaluate Jack's expected gain from this bet. [2 marks]

5. In a study of the effect of a particular drug on response times, 100 rats were each infected with a fixed dose of the drug. Each rat was then subjected to a stimulus and its response time (in seconds) recorded. The sample mean and standard deviation were calculated to be $\overline{X} = 1.16$ seconds and $\hat{\sigma} = 0.16$ seconds, respectively. Assuming that the response times are normally distributed, calculate a 95% confidence interval for the mean.

[5 marks]

The mean response time for rats who have not received the drug is known to be 1.2 seconds. On the basis of the confidence interval which you have calculated, comment on whether you believe the mean response time for rats who have received the drug differs from 1.2 seconds. [3 marks]

Section B

6. A market research firm was commissioned to secure data pertinent to the pricing of a new consumer product. A random sample of 25 potential consumers was interviewed and the figures below represent the maximum price at which each consumer would consider purchasing the product:

Maximum price (coded) at which purchase of a new product considered by each of 25 potential consumers:

7.56	9.58	9.59	9.62	9.66
9.80	9.81	9.81	9.83	9.85
9.90	10.02	10.02	10.05	10.10
10.17	10.19	10.29	10.32	10.37
10.51	10.55	10.77	10.86	15.97

a) Construct a frequency distribution of the data using the following class intervals:

7.5 - 9.5

9.5 - 10.0

10.0 - 10.5

10.5 - 11.0

11.0 - 16.0 [5 marks]

b) Find the mean price a randomly selected consumer would be willing to pay [2 marks]

c) Plot the corresponding cumulative frequency curve and provide an interpretation of this graph. [5 marks]

d) Let

y = (price x number of potential buyers)/25 denote the likely revenue per potential consumer.

Evaluate the values of y under the following two scenarios:

i) price = mean price evaluated in b) above,

ii) price = the second smallest price a consumer is willing to pay in the sample given above. [6 marks]

Comment on the behaviour of y in the two scenarios listed above. [2 marks]

7. a) A and B are two events in a probability space Ω and $\overline{A}$ and $\overline{B}$ are their respective complements. Interpret the set $\{(A \cap \overline{B}) \cup (\overline{A} \cap B)\}$ and prove that

$$P(\{(A \cap \overline{B}) \cup (\overline{A} \cap B)\}) = P(A) + P(B) - 2P(A \cap B).$$

[5 marks]

b) Two designs are being considered for a space vehicle. The first uses two main engines and the second four main engines during the critical period of launch. The first design relies on both engines functioning correctly, whereas for the second design the launch will be successful if at least three of the engines function correctly. The engines are of basically similar design and performance characteristics. In tests, the probability of failure for any particular engine is estimated as p, $0 < p < 1$, and it may be assumed that for both designs, engine failures occur independently.

i) For the first design, show that the probability that the launch is successful is given by $(1-p)^2$ [4 marks]

ii) For the second design, show that the probability that the launch is successful is given by $(1-p)^4 + 4p(1-p)^3$ [5 marks]

iii) Show that if $p > 2/3$, the first design is more likely to be successful at launch stage than the second design. [6 marks]

8. The time taken, X, by a motor to start after it has been switched on is a continuous random variable with probability density function:

$$f(x) = kx^{-2}, \quad 1 \le x \le 3,$$

where k is a constant.

Also,

$$f(x) = 0, \quad x < 1 \text{ or } x > 3$$

I) Show that:

a) $k = 1.5$; [3 marks]

b) $P(X < 1.5) = 0.5$; [3 marks]

c) $P(X > 2.5) = 0.1$. [3 marks]

II) Evaluate $E(X)$ and $V(X)$. [6 marks]

III) If the motor takes more than 1.5 seconds to start, a light comes on and stays on until either one second has elapsed or until the motor starts, whichever happens first. Let the random variable Y denote the length of time the light stays on.

Find

a) $P(Y=0)$; [2 marks]

b) $P(Y=1)$. [3 marks]

9. Suppose that the diastolic blood pressure, X, in a large population of hypertensive women has a normal distribution with mean 100 mmHg and standard deviation 16 mmHg.

a) Find the probability that a randomly chosen woman from this population will have a blood pressure more than 132 mmHg. [3 marks]

By using either a suitable approximation, or by an exact method, give numerical answers for each of the following probabilities in b), c) and e).

b) Probability that there are exactly 2 women with blood pressure more that 132 mmHg out of a random sample of 100 hypertensive women. [5 marks]

c) Probability that the average blood pressure of those 100 hypertensive women selected will be more than 104 mmHg. [5 marks]

d) Suppose that 300 independent random samples, each of size 100, is taken from this population with replacement (that is, each sample has returned to the population before taking a new sample). Let $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{300}\}$ denote the corresponding sample means, If a histogram of these sample means is drawn, describe in words what distribution/density it would look like. You don't need to draw any pictures.

[2 marks]

e) Probability that there are at most 2 means out of these 300 sample means, $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{300}\}$, for which the average blood pressure will be more than 104 mmHg.

[5 marks]

10. Fifty 1m^2 quadrants are laid down in random positions on a newly-cleared piece of land. After a suitable interval, numbers of plants of a particular annual weed were recorded for each quadrant. Table 1 below shows numbers of quadrants with 0, 1, 2,… specimen of weed.

Table 1

Number of quadrants, f_r, with r plants of a weed species

r	0	1	2	3	4	5	6	7	8	9	>9
f_r	4	11	5	8	11	8	2	0	0	1	0

a) Use the data in Table 1 to calculate the mean number of plants per quadrant.

[2 marks]

A common belief is that the plants of the particular weed species grow randomly in the cleared piece pf land and for testing this hypothesis it is proposed to fit a Poisson distribution to the data.

b) Compute the expected number of quadrants, e_r with r plants, according to the proposed Poisson model. [7 marks]

c) Test whether the data follow a Poisson distribution. [N.B.: It is sufficient to calculate the contributions to χ^2 *correct* to 1 decimal place.] [7 marks]

d) Suggest at least **two** reasons why such data, if relating to a weed whose seeds are dispersed solely by wind, *might* not follow a Poisson distribution. [4 marks]