## MATH161 January 2004 Exam Solutions

All similar to seen exercises except for 6(a), 8(b) and 10(a), which are bookwork.

# SECTION A

1. (i) Stemplot:

| | | |
|---|---|---|
| 3 | 9 | 3\|9 represents \$39000. |
| 4 | 002237789 | |
| 5 | 135556 | |
| 6 | 122 | |
| 7 | 2 | |

Unimodal, right skew.

(i) Have $n = 20$, so median is observation number 10.5, median $= (49 + 51)/2 = 50$.
LQ is obs. number 5.25, so LQ $= 0.75 \times 42 + 0.25 \times 43 = 42.25$.
UQ is obs. number 15.75, so UQ $= 0.25 \times 55 + 0.75 \times 56 = 55.75$.
IQR $= 55.75 - 42.25 = 13.5$.

2. (i) $\sum p(x) = 18k$, so $k = 1/18$.

(ii) $P(X > 1) = 13k = 13/18$.

(iii) $E[X] = k(0 \times 3 + 1 \times 2 + 2 \times 1 + 3 \times 0 + 4 \times 2 + 5 \times 4 + 6 \times 6) = 68k = 68/18 = 34/9$.

(iv) $E\left[X^2\right] = k(0 \times 3 + 1 \times 2 + 4 \times 1 + 9 \times 0 + 16 \times 2 + 25 \times 4 + 36 \times 6) = 354/18 = 59/3$, so $\mathrm{Var}[X] = (59/3) - (34/9)^2 = 437/81$.

3. Network reliability $= P(A)P(B)P(C \cup (D \cap E))P(F)$
$= P(A)P(B)(P(C) + P(D)P(E) - P(C)P(D)P(E))P(F)$
$= 0.8 \times 0.8 \times (0.8 + 0.64 - 0.512) \times 0.8 = 0.475136$.

4. (i) $P(X = x) = \binom{10}{x} 0.02^x 0.98^{10-x}$.

(ii) $E[X] = 10 \times 0.02 = 0.2$.

(iii) $P(X = 0) = 0.98^{10} = 0.817$.

(iv) $P(X > 3) = 1 - P(X \leq 3) = 1 - 0.98^{10} - 10 \times 0.02 \times 0.98^9 - 45 \times 0.02^2 \times 0.98^8 - 120 \times 0.02^3 \times 0.98^7 = 0.00003051$.

5. Have $Z = (X - 9)/2 \sim N(0, 1)$.

(i) $P(X > 7) = P(Z > -1) = P(Z < 1) = 0.8413$.

(ii) $P(7 < X < 12) = P(-1 < Z < 1.5) = P(Z < 1.5) - P(Z < -1)$
$= P(Z < 1.5) - (1 - P(Z < 1)) = 0.9332 - 1 + 0.8413 = 0.7745$.

(iii) $P(X < a) = 0.3085 \Rightarrow P(Z < (a - 9)/2) = 0.3085$
$\Rightarrow P(Z < (9 - a)/2) = 1 - 0.3085 = 0.6915 \Rightarrow (9 - a)/2 = 0.5 \Rightarrow a = 8$.

## SECTION B

6. (a) $P(A|B) = P(A \cap B)/P(B)$.

   (b) $L$ = Lied, $C$ = Cheated.

   (i) $P(L) = P(L|C)P(C) + P(L|\bar{C})P(\bar{C}) = 0.3 \times 0.55 + 0.2 \times 0.45 = 0.255$.

   (ii) $P(C|L) = P(L|C)P(C)/P(L) = 0.3 \times 0.55/0.255 = 0.647$.

   (c) (i) $P(A \cap B) = P(A)P(B)$

   (ii) $P(A \cup B) = P(A) + P(B) - P(A)P(B)$

   (iii) $P(B \cup C) = P(B) + P(C)$

   (iv) $P(B \mid A) = P(B)$

   (v) $P(B \mid (C \cap A)) = 0$

   (vi) $P(C \cup (A \cap B)) = P(C) + P(A)P(B)$

7. (a) Under independence, expected values are

| | N | L | H | Total |
|---|---|---|---|---|
| CHF | 131.61 | 102.24 | 47.15 | 281 |
| No | 764.39 | 593.76 | 273.85 | 1632 |
| Total | 896 | 696 | 321 | 1913 |

So that

$$\begin{aligned} X^2 &= \frac{(131.61-146)^2}{131.61} + \frac{(102.24-106)^2}{102.24} + \frac{(47.15-29)^2}{47.15} \\ &+ \frac{(764.39-750)^2}{764.39} + \frac{(593.76-590)^2}{593.76} + \frac{(273.85-292)^2}{273.85} \\ &= 1.573 + 0.139 + 6.988 + 0.271 + 0.024 + 1.203 = 10.197 \end{aligned}$$

Degrees of freedom $= 2 \times 1 = 2$. From tables, $\chi_2^2(0.05) = 5.991$. Value of $X^2$ is larger than the critical value, so there is evidence at the 5% level to reject the hypothesis of independence between alcohol consumption level and CHF.

(b) For Poisson with mean 1.1, then $P(X = x) = 1.1^x \exp(-1.1)/x!$, so we have

| Number of breakdowns | 0 | 1 | 2 | 3 | $\geq 4$ |
|---|---|---|---|---|---|
| Observed frequency | 16 | 22 | 18 | 9 | 5 |
| Expected frequency | 23.3 | 25.6 | 14.1 | 5.2 | 1.8 |

So goodness-of-fit statistic is

$$\begin{aligned} X^2 &= \frac{(23.3-16)^2}{23.3} + \frac{(25.6-22)^2}{25.6} + \frac{(14.1-18)^2}{14.1} + \frac{(5.2-9)^2}{5.2} + \frac{(1.8-5)^2}{1.8} \\ &= 2.29 + 0.51 + 1.08 + 2.84 + 5.68 \;=\; 12.40 \end{aligned}$$

Compare with $\chi_4^2(0.05) = 9.488$, value of $X^2$ is larger than the critical value, so reject the null hypothesis. There is evidence at the 5% level to reject the hypothesis that the data come from a Poisson distribution with mean 1.1.

Estimating mean from the data, have $\bar{x} = (16\times0+22\times1+18\times2+9\times3+5\times4)/70 = 105/70 = 1.5$.

With this mean value, have

| Number of breakdowns | 0 | 1 | 2 | 3 | $\geq 4$ |
|---|---|---|---|---|---|
| Observed frequency | 16 | 22 | 18 | 9 | 5 |
| Expected frequency | 15.6 | 23.4 | 17.6 | 8.8 | 4.6 |

So goodness-of-fit statistic is

$$\begin{aligned} X^2 &= \frac{(15.6-16)^2}{15.6} + \frac{(23.4-22)^2}{23.4} + \frac{(17.6-18)^2}{17.6} + \frac{(8.8-9)^2}{8.8} + \frac{(4.6-5)^2}{4.6} \\ &= 0.01 + 0.09 + 0.01 + 0.01 + 0.04 \;=\; 0.15 \end{aligned}$$

Compare with $\chi_3^2(0.05) = 7.815$, value of $X^2$ is smaller than the critical value, so cannot reject the null hypothesis. There is not sufficient evidence at the 5% level to reject the hypothesis that the data come from a Poisson distribution with mean determined from the data.

8. (a) 95% CI is $\bar{x} \pm 1.96s/\sqrt{n} = 1.16 \pm 1.96 \times 0.16/\sqrt{100} = 1.16 \pm 0.03136 = [1.129, 1.191]$.

CI excludes 1.2, so there seems to be evidence at the 5% level that the mean response time for rats who have received the drug differs from 1.2.

(b) Null hypothesis $H_0$ is what one believes in the absence of evidence to the contrary, in the example of part (a) above would have $H_0 : \mu = 1.2$ where $\mu$ is the mean response time for rats who have received the drug.

Alternative hypothesis $H_1$ is the hypothesis that some effect exists differing from that assumed by $H_0$, so in above example $H_1 : \mu \neq 1.2$, look for evidence to reject $H_0$ in favour of $H_1$.

Type I error occurs when one incorrectly rejects $H_0$ in favour of $H_1$.

Type II error occurs when one incorrectly fails to reject $H_0$ in favour of $H_1$.

Significance level is the probability of Type I error, given that $H_0$ is true.

Power is the probability of correctly rejecting $H_0$ given that $H_1$ is true.

(c) If $\mu = 1.2$ and $\sigma = 0.16$, then $(\bar{x} - \mu)/(\sigma/\sqrt{n}) = (1.16 - 1.2)/(0.16/10) = -2.5$, so with $Z \sim N(0, 1)$ the $p$-value for the test is

$p = P(Z < -2.5) + P(Z > 2.5) = 2(1 - P(Z < 2.5)) = 2(1 - 0.99379) = 0.01242$.

Less than 0.05, so at the 5% level reject $H_0$. There is evidence that the mean response time for rats who have received the drug differs from 1.2.

9. (a) (i) $\int_0^2 f(t)dt = \int_0^2 K\,(t^3 - 4t^2 + 5t)\,dt = K\left[(t^4/4) - (4t^3/3) + (5t^2/2)\right]_0^2$
$= K\,(4 - (32/3) + 10) = (10/3)K$, so that $K = 3/10$.

(ii) $E[T] = K\int_0^2 (t^4 - 4t^3 + 5t^2)\,dt = (3/10)\left[(t^5/5) - t^4 + (5t^3/3)\right]_0^2$
$= (3/10)\,((32/5) - 16 + (40/3)) = (3/10)(56/15) = 28/25 = 1.12.$

(iii) $E\left[T^2\right] = K\int_0^2 (t^5 - 4t^4 + 5t^3)\,dt = (3/10)\left[(t^6/6) - (4t^5/5) + (5t^4/4)\right]_0^2$
$= (3/10)\,((32/3) - (128/5) + 20) = (3/10)(76/15) = 38/25 = 1.52,$
so $\text{Var}[T] = (38/25) - (28/25)^2 = 166/625 = 0.2656$.

(b) For one car,
$P(T \le 1) = \int_0^1 K\,(t^3 - 4t^2 + 5t)\,dt = (3/10)\left[(t^4/4) - (4t^3/3) + (5t^2/2)\right]_0^1$
$= (3/10)\,((1/4) - (4/3) + (5/2)) = (3/10)(17/12) = 17/40 = 0.425,$
so the probability that none of the 10 cars spends more than 1 hour is
$0.425^{10} = 0.000192$.

(c) Have from above that $P(T \le t) = (1/40)\,(3t^4 - 16t^3 + 30t^2)$, so

$$\begin{aligned}
P(T \le 0.5) &= (1/40)\,((3/16) - 2 + (15/2)) = (1/40)(91/16) = 91/640 = 0.1421875 \\
P(0.5 < T \le 1) &= (17/40) - (91/640) = 181/640 = 0.2828125 \\
P(1 < T \le 1.5) &= (1/40)\,((243/16) - 54 + (135/2)) - (17/40) \\
&= (1/40)(459/16) - (181/640) = (459/640) - (17/40) \\
&= 187/640 = 0.2921875 \\
P(1.5 < T \le 2) &= 1 - (459/640) = 181/640 = 0.2828125
\end{aligned}$$

Probability mass function of $X$ is

$$\begin{aligned}
P(X = 0.4) &= 91/640, \\
P(X = 0.8) &= 181/640, \\
P(X = 1.2) &= 187/640, \\
P(X = 1.6) &= 181/640.
\end{aligned}$$

$E[X] = (0.4 \times 91 + 0.8 \times 181 + 1.2 \times 187 + 1.6 \times 181)/640 = 695.2/640 = £1.08625.$

10. (a) Binomial probabilities

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n$$

so expectation is

$$\begin{aligned}
E[X] &= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \left( \frac{n!}{x!\,(n-x)!} \right) p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} \left( \frac{n!}{(x-1)!\,(n-x)!} \right) p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \left( \frac{(n-1)!}{(x-1)!\,(n-x)!} \right) p^{x-1} (1-p)^{n-x} \\
&= np \sum_{y=0}^{n-1} \left( \frac{(n-1)!}{y!\,(n-1-y)!} \right) p^{y} (1-p)^{n-1-y} \qquad (y = x-1) \\
&= np \sum_{y=0}^{n-1} P(Y = y) \text{ where } Y \sim \text{Bin}(n-1, p) \\
&= np \times 1 = np.
\end{aligned}$$

Variance is $np(1-p)$.

(b) Number of Caesarian births in the sample $X \sim \text{Binomial}(50, 0.22)$, so

$$\begin{aligned}
P(11 \le X \le 14) &= \binom{50}{11} 0.22^{11} 0.78^{39} + \binom{50}{12} 0.22^{12} 0.78^{38} \\
&\quad + \binom{50}{13} 0.22^{13} 0.78^{37} + \binom{50}{14} 0.22^{14} 0.78^{36} \\
&= 0.1351 + 0.1238 + 0.1021 + 0.0761 = 0.4372
\end{aligned}$$

Have $E[X] = 50 \times 0.22 = 11$ and $\text{Variance}[X] = 50 \times 0.22 \times 0.78 = 8.58$, so

$$\begin{aligned}
P(11 \le X \le 14) &= P(10.5 \le X \le 14.5) \\
&\approx P\left( \frac{10.5 - 11}{\sqrt{8.58}} \le Z \le \frac{14.5 - 11}{\sqrt{8.58}} \right) \text{ where } Z \sim N(0, 1) \\
&= P(-0.17 \le Z \le 1.19) \\
&= P(Z \le 1.19) - P(Z \le -0.17) \\
&= P(Z \le 1.19) - (1 - P(Z \le 0.17)) \\
&= 0.8830 - 1 + 0.5675 \\
&= 0.4505
\end{aligned}$$

Approximation seems reasonably good, 0.4505 isn't far from 0.4372. Normal approximation to Binomial is good provided $np > 5$ and $n(1-p) > 5$. In this example, $np = 11$ and $n(1-p) = 39$, so would expect approximation to be good.