MATH101 January 2003 Exam Solutions

All similar to seen exercises except for 10(a), which is bookwork.

SECTION A

1. (i) Stemplot of test mark data:

4	1
5	
6	035
7	3455678
8	0123446667889
9	000135

4|1 represents 41%.

(ii) The mark distribution is unimodal, left skew, with a single outlying value well below the rest of the data at 41%.

(iii) Median is observation 15.5 = (83 + 84)/2 = 83.5. Would prefer median to mean, since the distribution is markedly skewed.

2. (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(ii) For mutually exclusive events, $P(C \cup D \cup E) = P(C) + P(D) + P(E)$.

(iii) G, H independent and F, G mutually exclusive, so
$P(F \cup (G \cap H)) = P(F) + P(G \cap H) = P(F) + P(G)P(H)$.

3. (i) $\int_0^2 K\,(9 + 6x - 3x^2)\,dx = K\left[9x + 3x^2 - x^3\right]_0^2 = K\,(18 + 12 - 8) = 22K$, so that $K = 1/22$.

(ii) $E[X] = \int_0^2 Kx\,(9 + 6x - 3x^2)\,dx = K\left[(9/2)x^2 + 2x^3 - (3/4)x^4\right]_0^2 = (1/22)\,(18 + 16 - 12) = 1$.

(iii) $E\left[X^2\right] = \int_0^2 Kx^2\,(9 + 6x - 3x^2)\,dx = K\left[3x^3 + (3/2)x^4 - (3/5)x^5\right]_0^2 = (1/22)\,(24 + 24 - (96/5)) = (1/22)(144/5) = 72/55$, so that $\text{Variance}[X] = (72/55) - 1^2 = 17/55$.

4. (i) $P(X = x) = \left(\frac{7^x}{x!}\right) \mathrm{e}^{-7}$.

(ii) $P(X = 10) = \left(\frac{7^{10}}{10!}\right) \mathrm{e}^{-7} = 0.071$.

(iii) $P(X \le 2) = \left(\frac{7^0}{0!} + \frac{7^1}{1!} + \frac{7^2}{2!}\right) \mathrm{e}^{-7} = 0.030$.

(iv) $P(X \le 1 \mid X \le 2) = \frac{P(X \le 1 \cap X \le 2)}{P(X \le 2)} = \frac{P(X \le 1)}{P(X \le 2)} = \frac{((7^0/0!)+(7^1/1!))\mathrm{e}^{-7}}{((7^0/0!)+(7^1/1!)+(7^2/2!))\mathrm{e}^{-7}} = \frac{1+7}{1+7+(49/2)} = 16/65$.

5. Have $Z = (X - 20)/4 \sim N(0, 1)$.

(i) $P(X > 20) = P(Z > 0) = 0.5$.

(ii) $P(14 < X < 22) = P(-1.5 < Z < 0.5) = P(Z < 0.5) - P(Z < -1.5) = P(Z < 0.5) - (1 - P(Z < 1.5)) = 0.6915 - 1 + 0.9332 = 0.6247$.

(iii) $P(X > a) = 0.2266 \Rightarrow P(Z > (a - 20)/4) = 0.2266 \Rightarrow P(Z < (a - 20)/4) = 1 - 0.2266 = 0.7734 \Rightarrow (a - 20)/4 = 0.75 \Rightarrow a = 23$.

SECTION B

6. (a) Given $P(A) = 0.4$, $P(B) = 0.4$, $P(C) = 0.2$ and $P(D|A) = 0.05$, $P(D|B) = 0.04$, $P(D|C) = 0.06$.

   (i) $P(D) = P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C) = 0.4 \times 0.05 + 0.4 \times 0.04 + 0.2 \times 0.06 = 0.048$.

   (ii) $P(B|D) = \dfrac{P(D|B)P(B)}{P(D)} = \dfrac{0.04 \times 0.4}{0.048} \approx 0.333$.

   (iii) $P(B|\bar{D}) = \dfrac{P(\bar{D}|B)P(B)}{P(\bar{D})} = \dfrac{0.96 \times 0.4}{0.952} \approx 0.4034$.

   (b) Probability a signal gets through components F, G is $0.9 \times 0.9 = 0.81$.

   Probability a signal gets through the part of the network made up of E, F, G, H, I is $0.8 \times (0.8 + 0.81 - 0.8 \times 0.81) \times 0.8 = 0.61568$.

   Probability a signal gets through the part of the network made up of A, B, C, D is $0.9 \times (0.9 + 0.8 - 0.9 \times 0.8) \times 0.8 = 0.7056$.

   Network reliability is $0.7056 \times 0.61568 \approx 0.4344$.

7. (a) For binomial with $p = 0.5$, with X denoting number of male children, then $P(X = x) = \binom{4}{x} 0.5^4$, so we have

Number of male children	0	1	2	3	4
Observed frequency	3	10	31	24	12
Expected frequency	5	20	30	20	5

So goodness-of-fit statistic is

$$\begin{aligned} X^2 &= \frac{(3-5)^2}{5} + \frac{(10-20)^2}{20} + \frac{(31-30)^2}{30} + \frac{(24-20)^2}{20} + \frac{(12-5)^2}{5} \\ &= 0.8 + 5 + 0.0333 + 0.8 + 9.8 \quad = \quad 16.433 \end{aligned}$$

Compare with $\chi_4^2(0.05) = 9.488$, value of X^2 is larger than the critical value, so reject the null hypothesis. There is evidence at the 5% level to reject the hypothesis that the data come from a binomial distribution with $p = 0.5$.

Estimating p from the data, have $\hat{p} = (3\times0+10\times1+31\times2+24\times3+12\times4)/(80\times4) = 192/320 = 0.6$.

Value of X^2 is given as 2.01; comparing with $\chi_3^2(0.05) = 7.815$, value of X^2 is smaller than the critical value, so cannot reject the null hypothesis. There is insufficient evidence at the 5% level to reject the hypothesis that the data come from a binomial distribution with $p = 0.6$.

(b) Under independence, expected values are

	S	M	L	Total
V	15.5	14	8.5	38
P	15.5	14	8.5	38
Total	31	28	17	76

So that

$$\begin{aligned} X^2 &= \frac{(15.5-6)^2}{15.5} + \frac{(14-20)^2}{14} + \frac{(8.5-12)^2}{8.5} + \frac{(15.5-25)^2}{15.5} + \frac{(14-8)^2}{14} + \frac{(8.5-5)^2}{8.5} \\ &= 2 \times (5.823 + 2.571 + 1.441) = 19.67 \end{aligned}$$

Degrees of freedom $= 2 \times 1 = 2$. From tables, $\chi_2^2(0.05) = 5.991$. Value of X^2 is larger than the critical value, so there is evidence at the 5% level to reject the hypothesis of independence between patient response and treatment.

3. (a) (i) If X is length of component, have $X \sim N(\mu, \sigma^2)$ with $P(X > 46.05) = 0.063$, $P(X < 45.95) = 0.025$.

Denote by Z a standard Normal random variable.

$P(Z > (46.05 - \mu)/\sigma) = 0.063 \Rightarrow P(Z < (46.05 - \mu)/\sigma) = 0.937 \Rightarrow (46.05 - \mu)/\sigma = 1.53$.

$P(Z < (45.95 - \mu)/\sigma) = 0.025 \Rightarrow P(Z < (\mu - 45.95)/\sigma) = 0.975 \Rightarrow (\mu - 45.95)/\sigma = 1.96$.

Adding: $(46.05 - 45.95)/\sigma = 3.49 \Rightarrow \sigma = 0.1/3.49 = 0.0287$

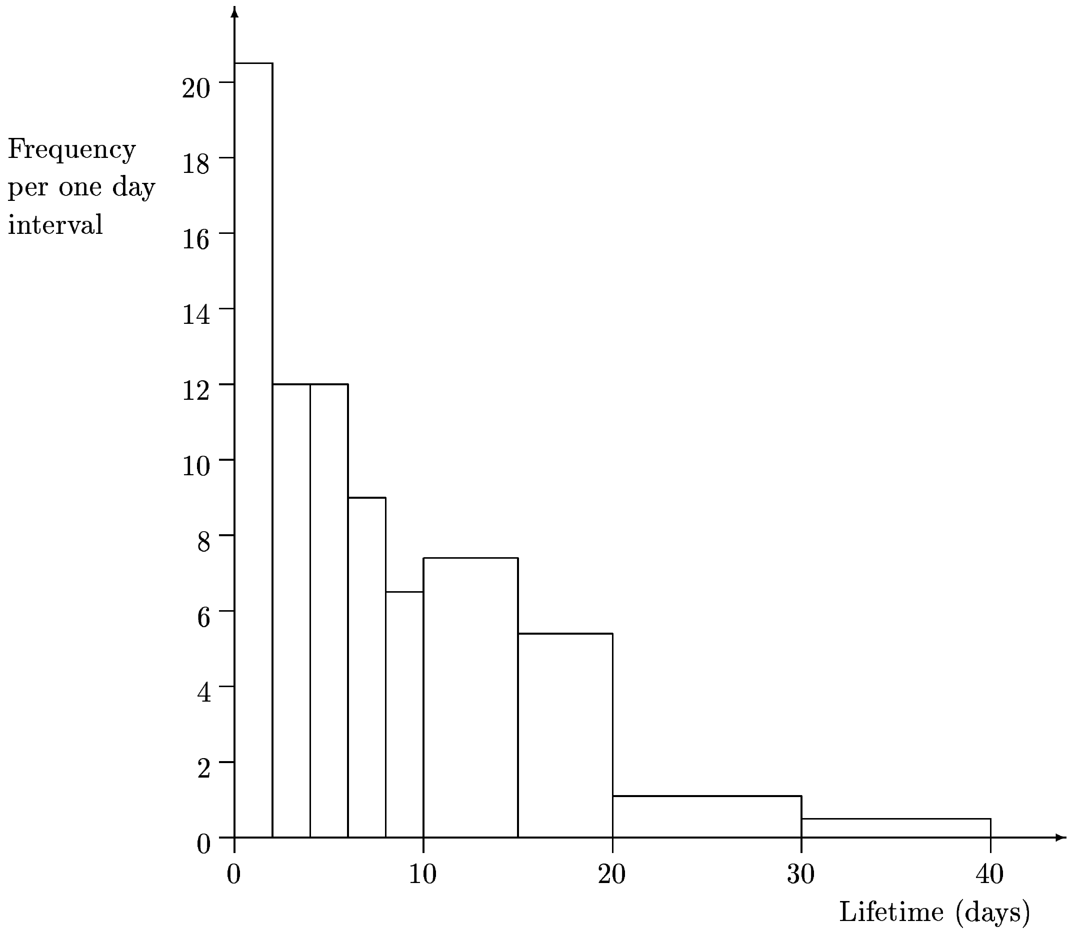
Subtracting: $(46.05 + 45.95 - 2\mu)/\sigma = -0.43 \Rightarrow 92 - 2\mu = -0.43\sigma \Rightarrow \mu = (92 + 0.43\sigma)/2 = 46.006$

(ii) In a sample of 5 components, let Y be number of acceptable length, then $Y \sim \text{Bin}(5, 0.912)$.

$$P(Y \geq 4) = 0.912^5 + 5 \times 0.912^4 \times 0.088 = 0.659 + 0.304 = 0.963.$$

(b) 95% CI is $\bar{x} \pm 1.96 s/\sqrt{n} = 104.25 \pm 1.96 \times 12.584/\sqrt{56} = 104.25 \pm 3.296 = [100.95, 107.56]$.
CI excludes 100, so there seems to be evidence at the 5% level that the mean PDI value for low-birthweight infants differs from 100.

9. (a) Histogram of component lifetime data.



Data clearly very skewed, so not Normally distributed.

(b) (i)

$$P(X < 10) = \int_0^{10} \lambda e^{-\lambda} \, dx = \left[-e^{-\lambda}\right]_0^{10} = 1 - e^{-10\lambda}$$

(ii) In part (a), 120 out of 200 components had lifetimes less than 10 days, so estimate the probability of a lifetime of less than 10 days as $120/200 = 0.6$.

(iii)

$$\begin{aligned} 1 - e^{-10\lambda} &= 0.6 \\ e^{-10\lambda} &= 0.4 \\ \lambda &= -\ln(0.4)/10 = 0.0916 \end{aligned}$$

Estimate $\hat{\lambda} = 0.0916$.

(iv)

$$\begin{aligned} \int_0^m 0.0916e^{-0.0916x} \, dx &= \left[-e^{-0.0916x}\right]_0^m = 1 - e^{-0.0916m} \\ \text{so } 1 - e^{0.0916m} &= 0.5 \\ e^{-0.0916m} &= 0.5 \\ m &= -\ln(0.5)/0.0916 = 7.565 \end{aligned}$$

10. (a) Poisson probabilities

$$P(X=x) \quad = \quad \frac{\lambda^x}{x!}\,\mathrm{e}^{-\lambda} \qquad\qquad x = 0, 1, 2, \ldots$$

so expectation is

$$\begin{aligned} E[X] &= \sum_{x=0}^{\infty} x P(X=x) = \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!}\,\mathrm{e}^{-\lambda} = \sum_{x=1}^{\infty} \lambda \frac{\lambda^{x-1}}{(x-1)!}\,\mathrm{e}^{-\lambda} \\ &= \lambda \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}\,\mathrm{e}^{-\lambda} = \lambda \sum_{x=0}^{\infty} P(X=x) = \lambda \times 1 = \lambda \end{aligned}$$

and variance is

$$\text{Variance}[X] \quad = \quad E\left[X^2\right] - \lambda^2 = E[X(X-1)] + \lambda - \lambda^2$$

where

$$\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1)P(X=x) = \sum_{x=0}^{\infty} x(x-1)\frac{\lambda^x}{x!}\,\mathrm{e}^{-\lambda} = \sum_{x=2}^{\infty} \lambda^2 \frac{\lambda^{x-2}}{(x-2)!}\,\mathrm{e}^{-\lambda} \\ &= \lambda^2 \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}\,\mathrm{e}^{-\lambda} = \lambda^2 \sum_{x=0}^{\infty} P(X=x) = \lambda^2 \times 1 = \lambda^2 \end{aligned}$$

so that

$$\text{Variance}[X] \quad = \quad \lambda^2 + \lambda - \lambda^2 = \lambda.$$

(b) Number of breakdowns $X \sim$ Poisson(3), so

$$\begin{aligned} P(X \geq 4) &= 1 - P(X \leq 3) \\ &= 1 - \left(1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!}\right)\mathrm{e}^{-3} \\ &= 1 - 13\mathrm{e}^{-3} = 1 - 0.6472 = 0.3528 \end{aligned}$$

Over a 10 week period, mean number of breakdowns is 30.

Poisson distribution with mean 30 can be approximated by $N(30, 30)$, so denoting by Y the number of breakdowns in 10 weeks and by Z a standard Normal random variable,

$$\begin{aligned} P(Y \geq 40) &= P(Y \geq 39.5) \\ &\approx P\left(Z \geq \frac{39.5 - 30}{\sqrt{30}}\right) \\ &= P(Z \geq 1.73) \\ &= 1 - P(Z < 1.73) = 1 - 0.9582 = 0.0418 \end{aligned}$$