PAPER CODE NO.
MATH 122

THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2005 EXAMINATIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Mathematics : Year 1
Master of Mathematics : Year 2
Master of Physics : Year 1

DYNAMIC MODELLING

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.
Take $g = 9.81\text{ms}^{-2}$. Give numerical answers to 3 significant figures.
You may use

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

SECTION A

1. A radioactive isotope disintegrates at a rate proportional to the amount present. Let $Q(t)$ be the amount of the isotope present (measured in milligrams) at any time t (where t is measured in days). Write down a differential equation for the rate of change dQ/dt of Q with time. Suppose initially 100mg of Q is present and after 1 week (7 days) this is reduced to 66.7mg. From your differential equation, find an expression for $Q(t)$ at any time. Also find the time interval that must elapse to reduce the amount present by one-half its original value.

[6 marks]

2. A swarm of locusts increases at a rate of 5% per hour. A powerful insecticide, sprayed by aeroplane, is capable of killing 10,000 per hour. Originally there were estimated to be 100,000 locusts in the swarm. If $n(t)$ represents the number of locusts at any time t, show that T, the time taken to kill off all the locusts, satisfies

$$T = 20\int_{0}^{100000} \frac{1}{200000 - n}\, dn .$$

Hence show that it takes approximately 14 hours to eradicate the swarm.

[7 marks]

3. John invests £250 per year in an account starting in year 0. Thus in year 1 and subsequent years he adds £250 to his account. The interest in the account is 4% per annum, which is also added to the account. The equation for the amount s_n in the account in year n in terms of s_{n-1}, the balance at the end of the previous year, is given by

$$s_n = 1.04 s_{n-1} + 250 \text{ pounds.}$$

Explain how this equation is derived. If $x_n = s_n - N$, where N is the equilibrium solution of this equation, write down a second equation for x_n in terms of x_0. Hence, or otherwise, calculate the amount in the account after 20 years.

[7 marks]

4. If n the number of events that have occurred at time t, follows a Poisson process, then the probability $P(n,t)$ that n events have occurred by time t is given by

$$P(n,t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

where λ is a constant. At what time does $P(n,t)$ reach its maximum value. If, on average, 12 mathematics students graduate with first class honours degrees from Liverpool University per year, calculate (assuming the number of first class honours maths students follows a Poisson process) the probability that this year there will be 15.

[5 marks]

5. A particle travels so that its velocity is given by

$$v = \cos(t) i + \sin(t) j + t k .$$

Find a) the particle's acceleration and b) the particle's position at any time t, given that at time $t = 0$ the particle was situated at $(0,0,1)$.

[6 marks]

6. A car travels in a straight line and exerts a constant driving thrust of T newtons. The earth exerts a resistive force equivalent to $(1 - e^{-\alpha t})T$ newtons, where t is the time in seconds after the start from rest and α is a positive constant. Write down the equation of motion for the car. Show that its velocity at any time t is

$$\frac{T}{m\alpha}\left(1 - e^{-\alpha t}\right)$$

metres per second, where m is the car's mass in kilograms. Find the distance travelled in this time.

[7 marks]

7. The equation for the displacement x of a forced harmonic oscillator is

$$\ddot{x} + 16x = 15\cos(t).$$

At time $t = 0$, $x = 3/2$ and $\dot{x} = 0$. Solve this equation to find x in terms of t. Find the value of $\dot{x}$ after $t = \pi/2$ seconds.

[8 marks]

8. Let $x(t)$ and $y(t)$ represent the levels of two populations governed by the following coupled differential equations

$$\frac{dx}{dt} = (20 - y), \qquad \frac{dy}{dt} = 4(x - 10).$$

At $t = 0$, $x = 10$ and $y = 24$. Obtain and solve the differential equation for y in terms of x. From your results draw the phase diagram for this situation, indicating which way around the curve the point (x, y) moves.

[7 marks]

SECTION B

9. Consider a two-state stochastic system, with states A and B. In the usual notation,

$$\frac{d}{dt}P(A,t)=P(B,t)W(B\to A)-P(A,t)W(A\to B).$$

Write down what each term represents. [5 marks]

A sixth form student wants to study at university, but cannot decide whether to do Chemistry or Maths. If on one day he prefers Maths, he changes his mind at a rate so that the probability that next day he chooses Maths is 0.5. If, however Chemistry is his choice on one day, the probability that the next day he chooses Chemistry is 0.4. Treating this problem as a two-state process, with $P(M,t)$ and $P(C,t)$ as the probabilities that at time t (in days) he chooses Maths and Chemistry respectively, show that

$$\frac{dP(M,t)}{dt}=0.6-1.1P(M,t).$$

Solve this equation to find $P(M,t)$ given at time $t=0, P(C,0)=0$. In the long term which subject is he most likely to do at university? [10 marks]

10. In the Canadian Wilderness a pack of wolves survives by hunting a reindeer herd. Let $x(t)$ represent the number of wolves in the pack and $y(t)$ the number of reindeer at time t months. The populations satisfy the differential equations

$$\frac{dx}{dt}=200-y, \qquad \frac{dy}{dt}=x-18, \qquad (x>0,\ y>0)$$

and initially it is estimated the wolf population was 13 and the reindeer 212. Obtain and solve the differential equation for y in terms of x. Deduce that the point $(x,\ y)$ lies on a circle of radius 13. Sketch this circle and show the direction in which this point moves. [7 marks]

By differentiating the first equation and substituting for dy/dt, show that

$$\frac{d^2x}{dt^2}+x=18.$$

Show that $dx/dt=-12$ when $t=0$. Solve this equation to find x as a function of t. Estimate the number of wolves in the pack in 12 months time. [8 marks]

11. Suppose the following differential equation

$$\frac{dn}{dt} = f(n),$$

has an equilibrium point at $n = N$ such that $f(N) = 0$. By considering what happens at $n = N + \varepsilon$, where ε is a small time dependent parameter, establish the conditions on $f(n)$ which determine the stability of the equilibrium point at $n = N$.

[5 marks]

Determine the equilibrium points of the following differential equation and their stability:

$$\frac{dn}{dt} = -n^2 + 100n - 1875.$$

Integrate the above equation (use partial fractions) to find $n(t)$, assuming $n = 50$ when $t = 0$. What happens to the value of n as $t \to \infty$?

[10 marks]

12. A cricket ball is thrown from a height of 2m with an initial speed of $\sqrt{74g}$ ms^{-1} at an angle of $\arctan(1/6)$ above the horizontal. Write down two second order differential equations for the horizontal and vertical distances, x and y travelled by the ball in *t* seconds. Solve these equations to find x and y as functions of *t*. Use these equations to show that the path taken by the ball is

$$y = 3 - \frac{1}{144}(x - 12)^2.$$

Find the maximum height reached by the ball and the horizontal distance travelled before its first bounce.

[15 marks]

13. At noon, an observer aboard a ship proceeding due West at $12\,\text{km h}^{-1}$ sees a second ship at a distance of 1 km due South of his ship. It appears to be travelling North East at $15\sqrt{2}\,\text{km h}^{-1}$.
By using vector methods, find the actual velocity and speed of the second ship.
[6 marks]
Find the position vector at time *t* of the second ship with respect to the first.
[3 marks]
Find the distance of closest approach and the time, to the nearest minute, at which this occurs.
[6 marks]

[Use unit vectors i and j to represent East and North respectively.]