PAPER CODE NO.
MATH 122

# THE UNIVERSITY *of* LIVERPOOL

## SEPTEMBER 2004 EXAMINATIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Mathematics : Year 1
Master of Mathematics : Year 2
Master of Physics : Year 1

DYNAMIC MODELLING

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.
Take $g = 9.81\text{ms}^{-2}$. Give numerical answers to 3 significant figures.
You may use

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

SECTION A

1. The average home crowd at Liverpool Football Club's Anfield ground is 45,000. If Liverpool lose, the crowd empties the ground at a rate of 9000 per minute. Write down the differential equation for $n(t)$, the number of fans in the ground at time *t* after the match. Solve this equation to find out how long it takes for the ground to empty.
If Liverpool win, fans linger to savour the victory, leaving at a new rate of $-750(t+1)$ per minute. Write down the differential equation for $n(t)$ in this case and find how long it takes for the ground to empty now.

[6 marks]

2. The supply of food for a certain population is subject to a seasonal change that affects the growth rate of the population. The differential equation

$$\frac{dn}{dt} = 2n\cos(2\pi t)$$

where $n(t)$ is the population level at time *t*, provides a simple model for the seasonal growth of the population. Solve the differential equation assuming the population level was $n_0$ at $t = 0$. Determine the maximum and minimum populations and the time interval between maxima.

[8 marks]

3. Jane's bank balance $b_n$ at the end of year *n* is given by

$$b_n = 0.97b_{n-1} + 2000 \qquad \text{pounds}$$

where $b_{n-1}$ is the balance at the end of the previous year. What is the equilibrium solution *N* of this equation? Defining the difference from equilibrium at the end of year *n* as $x_n = b_n - N$, show that $x_n = (0.97)^n x_0$. If Jane's balance at the end of year 0 is £2000, what is the balance to the nearest penny, at the end of year 10?

[6 marks]

4. If $n$ the number of events that have occurred at time $t$, follows a Poisson process, then the probability $P(n,t)$ that $n$ events have occurred by time $t$ is given by

$$P(n,t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

where $\lambda$ is a constant. Find the average value of $n$. If, on average, there are 15 jackpot lottery winners in Britain each year, calculate (assuming the number of jackpot winners follows a Poisson process) the probability that this year there 17.

[5 marks]

5. A particle travels so that its acceleration is given by

$$a = 6t\, i + \sin(3t)\, j + \cos(3t)\, k\ .$$

If at time $t = \pi/3$ the particle was situated at $(1,0,1)$ and travelling with velocity $4i + j - k$, find a) the velocity and b) the position of the particle at any time $t$.

[6 marks]

6. The engines of a railway locomotive of mass $m$ kg exert a constant force of $F$ N. The locomotive travels in a straight line with speed $v$ ms$^{-1}$, and as it moves it experiences a resistive force equivalent to $mkv$ N. Its initial speed was 0 ms$^{-1}$. Write down Newton's equation of motion. Show that

$$\frac{d}{dt}\left[e^{kt} v\right] = \frac{F}{m} e^{kt}\ .$$

Hence find the speed of the locomotive at time $t$.

[6 marks]

7. A particle of mass $m$ moves with velocity $v$ in a straight line along the $x$ axis under the influence of a position dependent force $F(x)$. Newton's equation of motion states

$$m\frac{dv}{dt} = F(x).$$

Using $dv/dt = v\,dv/dx$ find an expression for the particle's kinetic energy at any position $x$, given $v = 0$ at $x = a$. If $F(x) = -\lambda x$, where $\lambda > 0$ is a constant, show that the particle's maximum velocity $v_{Max}$ is given by

$$v_{Max} = \sqrt{\frac{\lambda}{m}}a\,.$$

[7 marks]

8. Two armies $X$ and $Y$ initially consisting of 50,000 and 75,000 men respectively meet on the battlefield. They fight until one or other army has been reduced to 5000 men. When this happens these remaining 5000 flee and the other army is declared the winner. Let $x(t)$ and $y(t)$ be the number of soldiers left alive in armies $X$ and $Y$ respectively after time $t$, and suppose the attrition rates satisfy the following differential equations

$$\frac{dx}{dt} = -10xy \qquad \frac{dy}{dx} = -16xy\,.$$

Draw the phase diagram associated with these equations, determine which army wins the battle and the total number of casualties on both sides.

[11 marks]

SECTION B

9. Explain what is meant by a stochastic process. [2 marks]
Consider a two-state stochastic system. At time *t*, let $P(A,t)$ be the probability the system is in state *A* and $P(B,t)$ the probability the system is in state *B*. Let $W(B \to A)$ be the probability per unit time the system goes from state *B* to *A* and let $W(A \to B)$ be the probability per unit time the system goes from state *A* to *B*. Write down an equation for $P(A, t + \delta t)$, the probability the system is in state *A* at time $t + \delta t$, and show that

$$\frac{d}{dt}P(A,t) = P(B,t)W(B \to A) - P(A,t)W(A \to B).$$

[5 marks]

A marksman enters a shooting contest firing pellets at a target. If he hits the bulls eye with a particular shot the probability he does so again is 0.98. By contrast the probability of him scoring a bulls eye after a miss is only 0.9. If $P(H,t)$ and $P(M,t)$ are the probabilities that he hits and misses the bulls eye respectively, show that

$$\frac{dP(H,t)}{dt} = 0.9 - 0.92P(H,t).$$

Solve this equation to find $P(H,t)$ given at the start of the competition $P(H,0) = 0.96$. In the long term to win the competition the marksman must hit the bulls eye at least 97.5% of the time. Is he likely to better this accuracy and have a chance of winning? [8 marks]

10. A spring of negligible mass and un-stretched length $L = 0.2\text{m}$, is stretched a distance 0.01m when suspended vertically from an armature and a mass *m* of 0.01kg is added to its free end. Calculate its spring constant $\lambda$. The mass is now set in motion by first pulling it down a further 0.02m and then releasing it from rest. Assuming $y = 0$ represents the position of the armature and $-y$ the position of the mass at any time *t*, write down Newton's equation of motion of the system. Show that the mass undergoes simple harmonic motion about the equilibrium point $y = y_0$ and that

$$y = A\cos\left(10\sqrt{g}t + \phi\right) + y_0.$$

Calculate the values of the constants $A$, $\phi$ and $y_0$. [10 marks]
Find an expression for the potential energy of the system when the mass is at any point $-y$ and hence show that the velocity of the mass *v* at any time is given by

[*Question continued on the next page*]

[*Question 10 continued*]

$$v = \sqrt{\frac{g\left[0.02^2 - (y - y_0)^2\right]}{m}}.$$

Hence find the maximum speed of the mass. How long after release does it attain this value? [Take $g = 9.81\text{ms}^{-1}$]

[5 marks]

11. Suppose the following differential equation

$$\frac{dn}{dt} = f(n),$$

has an equilibrium point at $n = N$ such that $f(N) = 0$. By considering what happens at $n = N + \varepsilon$, where $\varepsilon$ is a small time dependent parameter, establish the conditions on $f(n)$ which determine the stability of the equilibrium point at $n = N$.

[5 marks]

Determine the equilibrium points of the following differential equation and their stability.

$$\frac{dn}{dt} = -n^2 + 19n - 84$$

Integrate the above equation (use partial fractions) to find $n(t)$, assuming $n = 10$ when $t = 0$. What happens to the value of *n* as $t \to \infty$?

[10 marks]

12. A particle of mass $m$ is fired with initial speed $\mathrm{v}_0$ at an angle of $30^0$ to the horizontal from the origin $(x=0, y=0)$. Air resistance is directly proportional to the velocity of the particle, with constant of proportionality $\beta$. What are the units of $\beta$?

    Write down two differential equations for the horizontal and vertical forces acting on the body, in terms of $v_x$ and $v_y$ the horizontal and vertical components of the velocity of the particle. By means of an integration factor, integrate both equations and show the particle's speed at its maximum height is given by

    $$v_{Max} = \frac{\sqrt{3}\mathrm{v}_0 gm}{(\mathrm{v}_0\beta + 2gm)}.$$

    [10 marks]

    Integrate the equations again and show the horizontal range $R$ of the particle is given by

    $$R = \left(\frac{\sqrt{3}m\mathrm{v}_0 g t_F}{\mathrm{v}_0\beta + 2gm}\right),$$

    where $t_F$ is the time of flight.

    [5 marks]

13. A particle of mass m moves under the influence of a central force field of the form $F = F(r)\hat{r}$, where $\hat{r} = \cos(\theta)i + \sin(\theta)j$ is a unit vector in the direction of the line joining the centre of mass and the origin, and $\theta$ is the angle this line makes with the positive $x$ axis. Show that the angular momentum vector $L = m(r \times v)$, where $r = r\,\hat{r}$ is the position vector of the particle and $v = d\,r/dt$, is a constant vector. Find $v$ in terms of $(r,\theta)$ and their time derivatives, and hence show $L = mr^2\dot{\theta}\hat{k}$.

    [8 marks]

    Find the potential energy for a particle that moves in a force field $F = -\frac{K}{r^2}\hat{r}$.

    How much work is done by the force field in moving the particle from a point on the circle $r = a > 0$ to a point on the circle $r = b > 0$? Does the work done depend on the path? Explain.

    [7 marks]

 END