Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

Take $g = 9.81$ m s$^{-2}$. Give numerical answers to 3 significant figures.

You may use

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}.$$

## SECTION A

**1.** At noon, one lane of a three lane motorway becomes blocked and a queue starts to form. Sixty cars per minute join the queue and forty cars per minute leave the queue. Write down the differential equation for $n(t)$, where $n(t)$ is the number of cars in the queue at time $t$.

Solve this equation given that $n(0) = 0$.

After 45 minutes the blockage is cleared and 75 cars per minute leave the queue. Write down the new differential equation for $n(t)$. At what time does the queue clear?

[7 marks]

**2.** The temperature $T(t)$°C in a room being heated by a gas fire approximately obeys the differential equation

$$dT/dt = 6 - T/4.$$

The initial temperature is 12°C. Find the temperature at time $t$. Show that $T < 24$°C. How long does it take for the temperature to reach 22°C? Give your answer to the nearest second. [7 marks]

**3.** Consider two one-way roads which cross at right angles. There are traffic lights at the intersection. Cars arrive at the rate of 4 per minute at one set of lights which are at green for a time $T_1$ mins. At the other set, cars arrive at the rate of 3 per minute and the green light lasts for $T_2$ mins. In addition both lights are red for one minute at the same time to allow pedestrians to cross. Traffic passes a green light at the rate of 10 cars per minute.

Show that to avoid congestion, $10T_1 \geq 4(T_1 + T_2 + 1)$ and $-3T_1 + 7T_2 \geq 3$. Shade the region in the $T_1 - T_2$ plane for the times satisfied by these inequalities. Deduce that $T_2$ has to be greater than 1 minute. [8 marks]

**4.** In a "repayment" mortgage, for an initial loan of £$u_0$ a Building Society charges 8% interest at the start of each year, calculated on the amount still owed at that time. During each year, a borrower repays £8,000 to the Building Society, so at the end of the $m^{th}$ year he owes £$u_m$.

Show that

$$p_{m+1} = 1.08 p_m,$$

where $p_m = u_m - 100000$. Deduce that $p_m = 1.08^m p_0$.

It takes 25 years to repay the mortgage. How much was the initial loan? Give your answer to the nearest £10. [8 marks]

**5.** At time $t$, the position vector of a point, P, on a fairground ride is given by

$$\mathbf{r}(t) = (2\cos t + \cos 4t)\mathbf{i} + (2\sin t + \sin 4t)\mathbf{j} \text{ m} .$$

Find its acceleration $\mathbf{f}$ at time $t$. Show that

$$|\mathbf{f}|^2 = 260 + 64\cos 3t \quad \text{m}^2\text{s}^{-4}.$$

Hence, or otherwise, find the maximum and minimum values of $|\mathbf{f}|$. [6 marks]

By writing

$$\mathbf{r}(t) = \mathbf{s}(t) + \cos 4t \; \mathbf{i} + \sin 4t \; \mathbf{j} \text{ m} ,$$

where $\mathbf{s}(t) = 2\cos t \; \mathbf{i} + 2\sin t \; \mathbf{j}$ m , deduce that P moves around one circle whose centre moves around another circle. Roughly sketch part of the path taken by P. [2 marks]

**6.** The equation for the displacement, $x$, of a damped harmonic oscillator is

$$\ddot{x} + 8\dot{x} + 25x = 0.$$

At time $t = 0$, $x = 1$ and $\dot{x} = -4$. Find the value of $x$ at time $t$. [5 marks]

**7.** A golf ball is hit from the tee at the origin with a speed of $\sqrt{196.2}$ ms$^{-1}$ at an angle of 45° to the horizontal. Write down the equations of motion for the horizontal coordinate $x$ and the vertical coordinate $y$ of the ball at time $t$.

Solve these equations to show that

$$y = 5 - \frac{(x-10)^2}{20} \quad \text{m}.$$

[8 marks]

**8.** On average, I receive 10 telephone calls per day. Assuming that this is a Poisson process, find the probability that I receive exactly 40 calls in 5 days.

[4 marks]

## SECTION B

**9.** Consider a two-state stochastic system, with states $A$ and $B$. In the usual notation,

$$\frac{\mathrm{d}}{\mathrm{d}t}P(A,t) = P(B,t)W(B \to A) - P(A,t)W(A \to B).$$

Write down what each term in this equation represents.

Suggest one example of a two-state stochastic system. [5 marks]

For a particular system $W(A \to B) = \frac{1}{3}$, $W(B \to B) = \frac{3}{4}$ and $P(A,0) = 1$. For this system, find $P(A,t)$. What is the value of your $P(A,t)$ for very large $t$? [10 marks]

**10.** Let $x(t)$ represent the number of foxes and $y(t)$ the number of rabbits in a particular population at time $t$ days. Suppose that these satisfy the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{4}(y - 30), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{4}(40 - x) \qquad (x > 0,\ y > 0).$$

Initially there are 40 foxes and 80 rabbits.

Obtain and solve the differential equation for $y$ in terms of $x$.

From your results draw a phase diagram for this situation, indicating which way around the curve the point $(x, y)$ moves. [8 marks]

Using your diagram, find the maximum number of foxes in the population.

Also, find the number of foxes in the population when there are no rabbits left. Write down the differential equation for $x(t)$ when there are no rabbits. Solve this equation and deduce that after a further 11 days there are no foxes left.

[7 marks]

**11.** The number, $n(t)$ thousand, of fish in a particular lake satisfies the differential equation $(n > 0)$,

$$\frac{dn}{dt} = f(n) = 12n - n^2 - 32 \quad \text{per year.}$$

What are the equilibrium values of $n$?

By sketching $f(n)$ as a function of $n$, indicate (by the use of arrows) which of these values is stable and which is unstable. [8 marks]

Show, by using the method of partial fractions or otherwise, that

$$\frac{n-8}{n-4} = Ae^{-4t},$$

where $A$ is a constant.

Given that there are 10 thousand fish at time $t = 0$, show that

$$n = 4 + \frac{12}{3 - e^{-4t}}.$$

[7 marks]

**12.** One end of a light elastic *string* of length $L$ and modulus $\lambda = 4mg$ is attached to a fixed point $O$ on a ceiling. A particle of mass $m$ is attached to the other end, $A$, of the string.

The system is in equilibrium. Find the distance $OA$.

The particle is pulled down a further distance $\frac{3}{4}L$ and at time $t = 0$ it is released. Air resistance is neglected. By considering the conservation of energy, or otherwise, show that in the subsequent motion the particle just touches the point $O$. [15 marks]

**13.** At noon, an observer aboard a ship proceeding due East at 16 km h$^{-1}$ sees a second ship at a distance of $\sqrt{2}$ km due North of his ship. It appears to be travelling South West at $8\sqrt{2}$ km h$^{-1}$.

By using vector methods, find the actual velocity and speed of the second ship.

Find the position vector at time $t$ of the second ship with respect to the first.

Find the distance of closest approach and the time, to the nearest minute, at which it occurs.

[Use the unit vectors $\mathbf{i}$ and $\mathbf{j}$ to represent East and North respectively.]

[15 marks]