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Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

Take $g = 9.81$ m s^{-2}. Give numerical answers to 3 significant figures.

You may use

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}.$$

SECTION A

1. The height, $h(t)$, of water in a leaking cylinder decreases at the rate of $2t+1$ cm.s^{-1}. Write down the differential equation for h. Initially the height is 30 cm. Solve the differential equation to find $h(t)$. At what time is the height zero? [5 marks]

2. The height of a particular tree at time t is h metres. The growth of this tree is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{1}{10}h = k,$$

where k is a constant.

Solve this equation. You may assume that at time $t = 0$, the tree started to grow from seed.

The maximum height to which the tree could grow is 40 m. Determine the value of k. Find how high (to the nearest metre) the tree is 15 years after it started growing.

Sketch a graph of h against t. [8 marks]

3. A rat population increases at the rate of 10% per month. As a result of an extermination programme, 120 rats per month are killed. If there are 800 rats initially, how long does it take to eradicate all rats? [7 marks]

4. In a "repayment" mortgage, for an initial loan of £60,000 a Building Society charges 12% interest at the start of each year, calculated on the amount still owed at that time. During each year, a borrower repays £8,000 to the Building Society, so at the end of the n^{th} year she owes £u_n.

Show that

$$u_{n+1} = 1.12u_n - 8000.$$

By considering $p_n = u_n - N$, where N is the equilibrium value of u_n, or otherwise, show that

$$u_n = \left(60000 - \frac{200000}{3}\right)(1.12)^n + \frac{200000}{3}.$$

How long (to the nearest year) will she take to repay her mortgage?

[8 marks]

5. At time t, the position vector of a particle is given by

$$\mathbf{r}(t) = (2+3t)\mathbf{i} + (4-2t^2)\mathbf{j} + (t+t^2)\mathbf{k} \text{ m} .$$

Find its (i) its velocity, (ii) its speed and (iii) its acceleration at time t.

Show that $\mathbf{r}(t)$ may be written

$$\mathbf{r}(t) = \mathbf{a} + t\mathbf{b} + t^2\mathbf{c},$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are constant vectors which you should write down. [6 marks]

Hence, or otherwise, show that the particle's motion lies in the plane

$$(\mathbf{r}-\mathbf{a})\cdot\mathbf{n} = 0,$$

where $\mathbf{n}$ should be expressed in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [2 marks]

6. The engines of a ship, travelling in a straight line with the constant speed 8 m s^{-1}, stop at $t = 0$. The ship (of mass m kg) then slows through the non-constant resistive force R so that Newton's equation of motion is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mkv^{\frac{2}{3}}.$$

Sketch R as a function of v.

Given that $k = 3/400$ (in appropriate units) show that the ship comes to rest after a time 800 s and a distance 1600 m. [9 marks]

7. The equation for the displacement, x, of a damped harmonic oscillator is

$$\ddot{x} + 6\dot{x} + 25x = 0.$$

At time $t = 0$, $x = 1$ and $\dot{x} = -3$. Find the value of x at time t. [6 marks]

8. On average, 10 e-mails per day arrive at my PC. Assuming that the arrival of e-mails is a Poisson process, find the probability that exactly 40 e-mails arrive in 5 days? [4 marks]

SECTION B

9(a). A hovercraft travelling horizontally in a straight line starts from rest and accelerates uniformly during the first 6 minutes of its journey when it covers 2 km. Find its acceleration during this period and show that after 6 minutes its speed is $\frac{2}{3}$ kilometres per minute. [3 marks]

Then it moves at constant velocity until it experiences a constant retardation which brings it to rest in a further distance of 4 km. Find the magnitude of this retardation. Find, also, the time taken during retardation. [2 marks]

The total journey time is 42 minutes. Find the total distance travelled.
[You may use the equations $v = u + ft$ and $v^2 = u^2 + 2fx$ for a particle moving with constant acceleration f along the $x-$axis.] [3 marks]

(b). At time t, the acceleration of a particle of mass m is given by

$$-4\cos(2t)\mathbf{i} - 4\sin(2t)\mathbf{j} \quad \text{m s}^{-2} .$$

Initially, it is at the point (1,0) m with a velocity $2\mathbf{j}$ m s^{-1}. Find its velocity, $\mathbf{v}$, and position vector $\mathbf{r}$ at time t. Deduce that the particle's speed is constant and that its acceleration is always in the opposite direction to its position vector.

[7 marks]

10(a). Each day a student uses a mobile phone to ring his girlfriend. Sometimes her phone rings but at other times he finds that her phone is in use and he gets the "engaged" tone. The probability that her phone rings on day t, $P(ring, t)$, satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P(ring, t) = 0.7P(use, t) - 0.8P(ring, t).$$

What do the numbers 0.7 and 0.8 represent? Given that $P(ring, 0) = 0$, solve this equation and find the probability that her phone rings on day 3. [7 marks]

(b). The number, $n(t) \times 10^3$, of fish in a particular lake satisfy the differential equation ($n > 0$),

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \mathrm{f}(n) = 10n - n^2 - 16 \quad \text{per year.}$$

What are the equilibrium values of n?

By sketching $\mathrm{f}(n)$ v n, indicate (by the use of arrows) which of these values is stable and which is unstable.

Briefly describe what happens to the number of fish in the long term in the following cases: (i) the initial number of fish was 4000, (ii) the initial number of fish was 20000. [8 marks]

11. Let $x(t)$ represent the number of hares and $y(t)$ the number of foxes in a particular population at time t months. In a simplified model, these satisfy the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 60 - y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - 60 \qquad (x > 0,\ y > 0).$$

Initially there are 100 foxes and 90 hares.

(i) Obtain and solve the differential equation for y in terms of x. Deduce that the point (x, y) lies on a circle of radius 50. Sketch this circle and show the direction in which this point moves. [7 marks]

(ii) By differentiating the first equation and substituting for $\frac{\mathrm{d}y}{\mathrm{d}t}$, show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = 60.$$

Show that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is equal to -40 when $t = 0$.

Solve this equation and to find x as a function of t. Hence, or otherwise, find the number of foxes at time t.

[8 marks]

12. A ball is thrown from a height 2 m with a speed u m.s^{-1}, at an angle θ to the horizontal. Write down the two second order differential equations for the horizontal and vertical distances, x and y respectively, travelled by the ball in t seconds. Solve these equations to find x and y as functions of t. Use these solutions to show that the path taken by the ball is

$$y(x) = 2 + x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2.$$

[8 marks]

Given $\theta = \frac{1}{4}\pi$, find the speed required for this ball to hit a target at a point 10 m away horizontally and at a height of 3 m. [7 marks]

13. The position vector for a particle moving in a plane perpendicular to $\mathbf{k}$ is $\mathbf{r} = r\hat{\mathbf{r}}$, where $\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

Given that $\underline{\hat{\theta}}$ is defined by

$$\underline{\hat{\theta}} = \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{\mathbf{r}}$$

express $\underline{\hat{\theta}}$ in terms of $\cos\theta$, $\sin\theta$, $\mathbf{i}$ and $\mathbf{j}$. Hence show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\underline{\hat{\theta}}.$$

Show that

$$\hat{\mathbf{r}} \times \underline{\hat{\theta}} = \mathbf{k}.$$

[6 marks]

Show that its velocity is given by

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\underline{\hat{\theta}}.$$

Find its speed and its angular momentum, $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$. [7 marks]

Hence show that

$$v^2 = \dot{r}^2 + \frac{L^2}{m^2r^2}.$$

[2 marks]