PAPER CODE NO.
MATH 122

THE UNIVERSITY
of LIVERPOOL

MAY 2007 EXAMINATIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Mathematics : Year 1
Master of Mathematics : Year 2
Master of Physics : Year 1

DYNAMIC MODELLING

TIME ALLOWED : **Two Hours and a Half**

INSTRUCTIONS TO CANDIDATES

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.
Take $g = 9.81\text{ms}^{-2}$. Give numerical answers to 3 significant figures.
You may use

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

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SECTION A

1. At the end of 2006 the UK population *n* (measured in millions) was 60 million and growing at a natural rate of 1% . However, it was estimated that the population was swelled by half a million immigrants from Eastern Europe arriving each year. Write down a differential equation for $n(t)$ the population at time *t* (measured in years, with $t = 0$ representing 2006) and show that

$$n(t) = 110e^{0.01t} - 50 \qquad \text{millions.}$$

In how many years can we expect the population to reach 70 million?

[7 marks]

2. A block of ice cream is removed from the freezing compartment of a freezer at a temperature of $-5^0 C$. The ice cream is too cold to eat straight away, and is left on the sideboard to thaw a little. The temperature of the ice cream $\theta(t)$ at time *t increases* at a rate proportional to the difference between T_R the ambient room temperature ($20^0 C$) and θ , with *positive* constant of proportionality *k*. Write down a differential equation for $\theta(t)$, and integrate it. If after 10 minutes the ice cream's temperature has risen to $-3^0 C$, show that

$$k = \frac{1}{10}\ln\left[\frac{T_R + 5}{T_R + 3}\right].$$

If the ideal temperature for the ice cream is $0^0 C$, in how many minutes will it be ready to eat?

[9 marks]

3. At the end of the *m*'th year, a borrower owes £u_m to a Building Society for the mortgage on his house, where u_m satisfies

$$u_m = 1.08u_{m-1} - c \quad \text{pounds,}$$

where *c* is his yearly repayment. What is the annual interest rate charged by the Building Society? Write down *N* the equilibrium solution of this equation and show that $x_m = (1.08)^m x_0$, where $x_m = u_m - N$. If the borrower originally borrowed £150,000 for his house and he wishes to pay off his mortgage in 25 years, calculate *c* and the total amount of money he will pay back over that time.

[9 marks]

4. If n the number of events that have occurred at time t, follows a Poisson process, then the probability $P(n,t)$ that n events have occurred by time t is given by

$$P(n,t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

where λ is a constant. Find the average value of n. A lottery is held each week in which you participate and buy a ticket. The chance of buying a winning ticket and getting a prize is $1/100$. Assuming the number of buying of lottery tickets follows a Poisson process, what is the probability that you will win two prizes during the year (52 weeks).

[4 marks]

5. A particle initially at rest and situated at the origin is subject to a time dependent force causing the particle to accelerate in a manner given by

$$\boldsymbol{a} = (1-4t)\boldsymbol{i} + \cos(2t)\boldsymbol{j} + e^{2t}\boldsymbol{k}.$$

Find a) the particle's velocity and b) the particle's position at any subsequent time t. [$\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the x, y and z directions respectively.]

[6 marks]

6. A horse pulls a plough of mass m kg in a straight line, exerting a force $[4+\sin(t/2)]$ Newtons at time t seconds. The resistance to the ploughs motion is $[4-\exp(-t/2)]$ Newtons. Write down Newton's equation of motion for the plough and use it to find both the velocity and the distance the plough has travelled by the time t seconds, given that it was initially at rest.

[6 marks]

7. The equation for the displacement x of a damped harmonic oscillator is

$$\ddot{x} + 8\dot{x} + 41x = 0$$

At time $t = 0$, $x = 1$ and $\dot{x} = 6$. Solve this equation to find x in terms of t. At what time $t > 0$ does x first fall to zero?

[7 marks]

8. Let $x(t)$ and $y(t)$ represent the concentrations at time t of two species of fish larvae competing against one another in a pond. The concentrations are governed by the following coupled differential equations

$$\frac{dx}{dt} = xy, \qquad \frac{dy}{dt} = y^2(2-x), \qquad (x>0,\ y>0).$$

At $t=0$, the respective concentrations are $x=1$ and $y=e^{-1}$ per millilitre. Obtain and solve the differential equation for y in terms of x. From your results draw the phase diagram for this situation, indicating which way along the curve the point (x, y) moves. Find the maximum concentration of the y species of larva.

[7 marks]

SECTION B

9. Explain what is meant by a stochastic process. [2 marks]
Consider a two-state stochastic system. At time t, let $P(A,t)$ be the probability that the system is in state A and $P(B,t)$ the probability that it is in state B. Let $W(B \to A)$ be the probability per unit time the system goes from state B to state A and let $W(A \to B)$ be the probability per unit time the system goes from state A to state B. Write down an equation for $P(A, t+\delta t)$, the probability the system is in state A at time $t+\delta t$, and show that

$$\frac{d}{dt}P(A,t) = P(B,t)W(B \to A) - P(A,t)W(A \to B).$$

[5 marks]

Every weekday a university lecturer must travel to work in Liverpool from Birkenhead. She travels either by ferry or catches the bus. If she goes to work by bus one day, the probability that she does so again the next day is 0.75. However, if she chooses to catch the ferry one particular day the probability that she switches to the bus the next is 0.2. If $P(B,t)$ and $P(F,t)$ are the probability she catches the bus or ferry respectively, show that

$$\frac{dP(F,t)}{dt} = 0.25 - 0.45P(F,t).$$

Solve this equation to find $P(F,t)$, given at time $t=0$, $P(F,0)=0.5$. In the long term what is the probability she catches the ferry? [8 marks]

10. Bird Flu infects both the wild bird population of the UK and the domestic poultry flocks destined for human consumption. Let $w(t)\times 10^6$ represent the number of wild birds infected at time *t* (in months) and $p(t)\times 10^6$ the number of domestic poultry infected at the same time. The spread of the disease can be modelled by the following couple differential equations

$$\frac{dp}{dt} = w - 23, \qquad \frac{dw}{dt} = \frac{25}{9}(7 - p), \qquad (p > 0, w > 0).$$

Initially it is estimated that 4 million domestic poultry are infected along with 23 million wild birds.
Obtain and solve the differential equation for *w* in terms of *p*. Deduce that the point (p, w) lies on an ellipse. Sketch this ellipse and show the direction in which this point moves. [7 marks]
By differentiating the first equation and substituting for dw/dt, show that

$$\frac{d^2p}{dt^2} + \frac{25p}{9} = \frac{175}{9}.$$

Show that $dp/dt = 0$ when $t = 0$. Solve this equation to find *p* as a function of *t*. If nothing is done to check the disease, estimate the number of birds infected in the poultry flock in $t = 2$ months time?

[8 marks]

11. The number of people queuing at a supermarket checkout $n(t)$ at time t, where *t* is measured in hours, satisfies the following differential equation

$$\frac{dn}{dt} = n^2 - 26n + 120.$$

Determine the equilibrium points of the differential equation and their stability. Plot a graph of dn/dt against *n* showing the equilibrium points and indicating the evolution of *n* in their vicinity. What happens to the queue if initially there are 3 people waiting to be served?

[8 marks]

Solve the differential equation to find $n(t)$ (use partial fractions), given that currently at $t = 0$, that there were 18 people waiting to be served. How long will it take (to the nearest second) for the queue to fall to just 9 people.

[7 marks]

12. An aeroplane of mass m kg lands and decelerates by means of two mechanisms. Firstly the pilot uses the engine as a brake, which applies a braking force equivalent to $3m$ Newtons. Secondly he lowers the flaps and by so doing increases the air resistance by an amount equivalent to $0.01mv$ Newtons, where v is the velocity of the plane on the runway. Its initial landing velocity is 100ms^{-1}. Write down Newton's Equation of Motion, and show that it takes about 29 seconds for the plane to come to a halt.

[7 marks]

Show that $dv/dt = v\,dv/dx$ where x is the distance the aeroplane has travelled along the runway before it comes to rest. Hence, or otherwise, find the distance the aeroplane travels along the runway before coming to rest.
[Hint $v/(3+0.01v) = 100 - 300/(3+0.01v)$.] [8 marks]

13. A man is lost in the desert and is slowly dying of thirst. Using the sun as guide he estimates he is walking at 4km/h due west. He initially observes what appears to be a Bedouin tribesman riding a camel some 0.6km due south of his position. The Bedouin appears to the man to be travelling north-west at a speed of 10km/h.
Using vector methods, calculate the actual velocity and speed of the Bedouin.

[6 marks]

Find the position vector at time t of the Bedouin with respect to the man.

[3 marks]

In the blistering heat the man is unsure whether the image of the Bedouin is just a mirage, and he needs to close to within 400m before he can hail the tribesman for assistance. Given their current speeds will he be able to reach such a position and establish the whether tribesman is real or a figment of his imagination?

[6 marks]

[Hint: Use unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ to represent East and North respectively.]