PAPER CODE NO.
MATH 122

THE UNIVERSITY
of LIVERPOOL

MAY 2006 EXAMINATIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Mathematics : Year 1
Master of Mathematics : Year 2
Master of Physics : Year 1

DYNAMIC MODELLING

TIME ALLOWED : **Two Hours and a Half**

INSTRUCTIONS TO CANDIDATES

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.
Take $g = 9.81\text{ms}^{-2}$. Give numerical answers to 3 significant figures.
You may use

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

SECTION A

1. The Indian tiger population grows at a rate of 2% per annum. However, poachers and local villagers kill, on average, 200 tigers per year, selling their skins on to traders in the Far East. If $n(t)$ represents the population of Indian tigers at time *t* (measured in years), write down a differential equation for the rate of change of *n*. If at the start of 2006 (equivalent to $t = 0$) there were 4000 Indian tigers left alive, integrate your differential equation and show that

$$n(t) = 10{,}000 - 6000e^{0.02t} .$$

How long will it be before the Indian tiger becomes extinct?

[7 marks]

2. A north-south main road has one lane each way. At present one lane is closed for 100 metres to allow the road to be resurfaced. Traffic lights have been installed so that traffic must use the second lane, alternating in each direction. Vehicles pass through a green light at the rate of 15 vehicles per minute. Vehicles going north arrive at one set of lights, which are green for T_1 minutes, at a rate of 3 per minute. Southbound vehicles arrive at another set of lights, which are green for T_2 minutes, at a rate of 10 per minute. In addition, *both* sets of lights are red for 1 minute at the same time, in order for the open lane to clear.
Show that to avoid congestion, $4T_1 \geq T_2 + 1$ and $T_2 \geq 2T_1 + 2$. Shade the region in the T_1 - T_2 plane for the times satisfied by these inequalities. Deduce that T_1 has to be at least one and a half minutes and find the minimum value of T_2 .

[9 marks]

3. My bank has offered me a personal loan of £10000 at an interest rate of 6.6% annum over 10 years. Show that this is equivalent to a rate of 0.534% per month. Write down the equation for u_{m+1}, the balance owed at the end of the $(m+1)$th month, in terms of u_m and *c*, the fixed amount I must repay each month. Let *N* be the equilibrium solution of this equation and $p_m = u_m - N$. *Write down* the equation for p_m in terms of p_0 . Given $u_{120} = 0$, show that

$$(N - 10000) = N/(1.00534)^{120} .$$

Solve this equation for *N* and hence, or otherwise, calculate *c* and the total amount I must pay back over the 10 years of the loan.

[8 marks]

4. If n the number of events that have occurred at time t, follows a Poisson process, then the probability $P(n,t)$ that n events have occurred by time t is given by

$$P(n,t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

where λ is a constant. Find the average value of n. If, on average, there are 2 earthquakes per week in the western United States, calculate (assuming the number of earthquakes follows a Poisson process) the probability that there will be exactly three earthquakes in the next two weeks.

[4 marks]

5. A particle travels so that its velocity is given by

$$\boldsymbol{v} = \cosh(t)\boldsymbol{i} + e^{-3t}\,\boldsymbol{j} + \frac{t^2}{2}\boldsymbol{k}\,.$$

Find a) the particle's acceleration and b) the particle's position at any time t, given that at time $t = 0$ the particle was situated at $(0, -1/3, 1/3)$.

[5 marks]

6. A Merseyrail train of mass m is travelling in a horizontal straight line at a constant speed of $16\mathrm{ms}^{-1}$. At time $t = 0$ the driver puts on the brakes to bring the train to a halt at the next station. The train's brakes exert a non constant resistive force equivalent to $-mv^{1/2}/6$ newtons, where v is the instantaneous velocity of the train. Write down Newton's equation of motion for this system and show the train comes to a halt after 48 seconds. Also show that the train travels just over $1/4\,\mathrm{km}$ after the brakes are applied before stopping.

[7 marks]

7. The equation for the displacement x of a forced harmonic oscillator is

$$\ddot{x} + 9x = 4\cos(t).$$

At time $t = 0$, $x = 1/2$ and $\dot{x} = 3/2$. Solve this equation to find x in terms of t.

[7 marks]

8. Let $x(t)$ and $y(t)$ represent the levels of two populations governed by the following coupled differential equations

$$\frac{dx}{dt} = (y - 23), \qquad \frac{dy}{dt} = \frac{25}{9}(7 - x).$$

At $t = 0$, $x = 7$ and $y = 18$. Obtain and solve the differential equation for y in terms of x. From your results draw the phase diagram for this situation, indicating which way around the curve the point (x, y) moves.

[8 marks]

SECTION B

9. Consider a two-state stochastic system, with states A and B. In the usual notation,

$$\frac{d}{dt}P(A,t) = P(B,t)W(B \to A) - P(A,t)W(A \to B).$$

Write down what each term represents. [5 marks]
The Government Passport Agency's computer is infested by software bugs that cause it to crash. If on one day it crashes, the probability that next day it also crashes is 0.4. However, if the computer runs perfectly on one day the probability that it crashes the next is only 0.2. Treating the problem as a two-state process, with $P(R,t)$ and $P(C,t)$ as the probabilities that at time t (in days) the computer runs and crashes respectively, show that

$$\frac{dP(R,t)}{dt} = 0.6 - 0.8P(R,t).$$

Solve this equation to find $P(R,t)$ given at time $t = 0$, when the computer was first installed, $P(R,0) = 0.5$. What is the value of $P(R,t)$ as $t \to \infty$?

[10 marks]

10. Before large scale European settlement, Native American tribes survived by hunting the buffalo across the Great Plains. Let $x(t)$ represent the number of Native Americans in one particular tribe and $y(t)$ the number of buffalo in the herd it hunts, where t represents time. The populations were estimated to satisfy the differential equations

$$\frac{dx}{dt} = \frac{(300 - y)}{16}, \qquad \frac{dy}{dt} = x - 40, \qquad (x > 0,\, y > 0)$$

and at $t = 0$ (during the height of summer) the populations were believed to be $x = 56$ Native Americans and $y = 348$ buffalo respectively.
Obtain and solve the differential equation for y in terms of x. Deduce that the point (x, y) lies on an ellipse. Sketch this ellipse and show the direction in which this point moves. [8 marks]
By differentiating the first equation and substituting for dy/dt, show that

$$\frac{d^2x}{dt^2} + \frac{x}{16} = \frac{5}{2}.$$

Show that $dx/dt = -3$ when $t = 0$. Solve this equation to find x as a function of t. Estimate the size of Native American tribe when $t = 4\pi$ (that is in mid winter in 6 months time).
[7 marks]

11. The population $n(t) \times 10^3$ of Southern Africa's Black Rhinoceros at time $t \times 10^{-2}$, where t is measured in years, is believed to satisfy the following differential equation

$$\frac{dn}{dt} = 10n - n^2 - 16.$$

Determine the equilibrium points of the differential equation and their stability. Plot a graph of dn/dt against n showing the equilibrium points and indicating the evolution of n in their vicinity. What is the future of the Black Rhinoceros if its population falls below 2000?
[8 marks]
Solve the differential equation to find $n(t)$ (use partial fractions), given that currently at $t = 0$, there are estimated to be 4000 Black Rhinos in Southern Africa. How many (to the nearest integer) are there likely to be in 10 years?
[7 marks]

12. A ball of mass m kg is thrown vertically upwards with an initial velocity of u ms^{-1}. It experiences an air resistance of mkv Newtons, when its velocity is v ms^{-1}, where k is a constant. Write down Newton's Equation of Motion. What are the units of k? Show that the ball reaches a maximum height h, where

$$h = \frac{u}{k} - \frac{g}{k^2} \ln\left[1 + \frac{ku}{g}\right].$$

[HINT: Use $dv/dt = v\,dv/dy$ where y is the ball's height above the ground.]

[9 marks]

Given that it takes a time T to reach this height, show that

$$\int_u^0 \frac{dv}{g + kv} = -\int_0^T dt\,.$$

Hence, or otherwise, find T and deduce that

$$hk + gT = u\,.$$

[6 marks]

13. Briefly explain the Conservation of Energy for a particle moving in one dimension. What is the associated potential energy $V(x)$ for a position dependent force of the form $F(x) = -kx$, where x is the displacement from some fixed point and k is a constant?

[3 marks]

A bungee jumper of mass 60 kg, initially at rest, drops from a crane 100m above the ground falling under the force of gravity. The *un-stretched* length of rope attaching him to the crane is 24m long. What is the jumper's velocity after falling this distance, i.e. immediately before the rope begins to stretch.

[5 marks]

The stretching rope then exerts an upward retarding force $-10g(x-24)$ Newtons, where x is the distance measured downwards from the crane. Show that for $x \geq 24$ the jumpers speed is given by

$$v^2 = 2g\left[x - \frac{1}{12}(x-24)^2\right].$$

What is the maximum distance that he falls?

[7 marks]