PAPER CODE NO.
MATH 122

THE UNIVERSITY
of LIVERPOOL

SUMMER 2005 EXAMINATIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Mathematics : Year 1
Master of Mathematics : Year 2
Master of Physics : Year 1

DYNAMIC MODELLING

TIME ALLOWED : **Two Hours and a Half**

INSTRUCTIONS TO CANDIDATES

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.
Take $g = 9.81\mathrm{ms}^{-2}$. Give numerical answers to 3 significant figures.
You may use

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

SECTION A

1. Newton's law of cooling is expressed mathematically as

$$\frac{dT}{dt} = -k\left(T - T_S\right),$$

where T is the temperature of the cooling body. What is T_S ? By defining $x = \left(T - T_S\right)$, determine the differential equation for x. Write down the solution of this equation.
A cake baked at $80^0 C$ is removed from the oven and placed on the kitchen sideboard to cool. If the kitchen temperature is a constant $20^0 C$, how long does it take for the cake to cool to $30^0 C$? [Assume $k = 2 \times 10^{-4} \mathrm{s}^{-1}$]

[6 marks]

2. My credit card company has offered me a loan of £4000 to be repaid over a period of 6 years at rate of 8% per annum. I wish to pay off the loan by paying a fixed amount c each *month*. Show that the annual rate of interest is equivalent to a monthly rate of 0.643%. Write down the equation for u_{m+1} , the balance owed at the end of month $\left(m+1\right)$ in terms of u_m and c. If $x_m = u_m - N$ where N is the equilibrium solution of this equation, write down a second equation for x_m in terms of x_0 . If after 6 years (72 months) $u_{72} = 0$ calculate the size of my monthly repayments c and the total amount I will pay during the loan period.

[7 marks]

3. Consider two one-way roads that cross at right angles. There are traffic lights at the intersection. Cars arrive at the rate of 3 per minute at one set of lights, which are green for a time T_1 mins. At the other set, cars arrive at a rate of 6 per minute and the green light lasts for T_2 mins. Both lights are red for one minute at the same time to allow pedestrians to cross. Traffic passes a green light at the rate of 10 cars per minute.
Show that to avoid congestion $7T_1 \geq 3T_2 + 3$ and $2T_2 - 3T_1 \geq 3$. Shade the region in the $T_1 - T_2$ plane for the times satisfied by these inequalities. If the maximum time the lights can be green is 90 seconds will congestion build up?

[7 marks]

4. On average I receive 12 E-mails per day. Assuming this is a Poisson process find the probability I receive exactly 36 E-mails in 3 days.

[5 marks]

5. At time t the position vector $\boldsymbol{r}(t)$ of a point, P, on a model fair ground ride is given by

$$\boldsymbol{r}(t) = [3\cos(t) + \cos(2t)]\boldsymbol{i} + [3\sin(t) + \sin(2t)]\boldsymbol{j}\ .$$

Find its acceleration $\boldsymbol{a}$ at any time t. Show that

$$|\boldsymbol{a}|^2 = 25 + 24\cos(t).$$

Hence or otherwise find the maximum and minimum values of $|\boldsymbol{a}|$.

[8 marks]

6. A particle of mass m moves in a straight line under the influence of a constant force F. Assuming that there is a resisting force numerically equal to kv^2, where v is the instantaneous speed and k is a constant, prove that the distance travelled in going from speed v_1 to v_2 is

$$\frac{m}{2k}\log\left|\frac{F - kv_1^2}{F - kv_2^2}\right|.$$

[Hint: $\dfrac{dv}{dt} = v\dfrac{dv}{dx}$.]

[8 marks]

7. The equation for the displacement x of a damped harmonic oscillator is

$$\ddot{x} + 2\dot{x} + 5x = 0\ .$$

At time $t = 0$, $x = 0$ and $\dot{x} = 6$. Find x at time t and sketch its graph.

[7 marks]

8. Let $x(t)$ and $y(t)$ represent the levels of two animal populations governed by the following coupled differential equations

$$\frac{dx}{dt} = \frac{1}{2}(y - 2000) \qquad \frac{dy}{dt} = 2(1000 - x).$$

Initially $x = 1040$ and $y = 1940$. Obtain and solve the differential equation for y in terms of x. From your results draw a phase diagram for this situation, indicating which way around the curve the point (x, y) moves.

[7 marks]

SECTION B

9. Explain what is meant by a stochastic process. [2 marks]

Consider a two-state stochastic system. At time t, let $P(A,t)$ be the probability that the system is in state A and $P(B,t)$ the probability that it is in state B. Let $W(B \to A)$ be the probability per unit time the system goes from state B to state A and let $W(A \to B)$ be the probability per unit time the system goes from state A to state B. Write down an equation for $P(A, t + \delta t)$, the probability the system is in state A at time $t + \delta t$, and show that

$$\frac{d}{dt} P(A,t) = P(B,t) W(B \to A) - P(A,t) W(A \to B).$$

[5 marks]

Once a week Mike likes to spend an evening either at the cinema or meeting friends in the pub. If he visits the cinema one week the probability he will do so again next week is 0.4. In contrast if he goes to the pub one week the probability he goes to the cinema the next week is 0.7. If $P(cin,t)$ and $P(pub,t)$ are the probabilities Mike goes to the cinema and pub respectively on week t, show that

$$\frac{dP(cin,t)}{dt} + 1.3P(cin,t) = 0.7 .$$

Solve this equation if $P(cin,0) = 0.5$. In the long term, how many times is Mike likely to go to the cinema during one year (52 weeks)?

[8 marks]

10. In London at the time of the Black Death in 1348 the city was infested with rats. The rat population $x(t)\times 10^4$ per unit area at time t was thought to grow at a rate

$$\frac{dx}{dt} = xy ,$$

where $y(t)\times 10^4$ is the human population at that time. The rats carry the bubonic plague virus which is lethal to humans. As a result the rate of change of the human population was estimated to satisfy

$$\frac{dy}{dt} = y^2(4-x).$$

Find the equation for dy/dx and integrate it, assuming that at $t=0$, $x=2$ and $y=1$.

[7 marks]

Sketch the graph of y against x, indicating the direction that x and y change with time. Describe what happens to the two populations. Calculate the maximum human population before the plague virus takes hold.

[8 marks]

11. The rate at which people join a queue is $10+9n-n^2$ per minute where $n(t)$ is the number of people in the queue at time t. If no one is leaving the queue, what would be the final number in the queue?

[2 marks]

However, people are leaving the queue at a rate of 28 per minute. Show that $n(t)$ satisfies the differential equation

$$\frac{dn}{dt} = -\left(n^2 - 9n + 18\right).$$

What are the possible equilibrium values of $n(t)$ and which ones are stable?

[6 marks]

Solve the differential equation to find $n(t)$ given there are four people in the queue at time $t=0$. [Hint: use partial fractions.]

[7 marks]

12. A golf ball of mass m is hit from ground level on a horizontal golf course with a speed u at an angle 45^0 to the horizontal. The only force acting on the ball during its flight is that due to gravity.
Write down the two second order differential equations for the horizontal and vertical distances x and y travelled by the ball in t seconds. Solve these equations to find x and y as functions of t. Find, in terms of u and g, the time taken to the first bounce, and show this occurs at a horizontal distance $D = u^2/g$.

[8 marks]

Later in the day the golfer repeats his shot only this time the wind has got up and, in addition to the force of gravity, the ball experiences a resistive force of $-mkv_x$ in the direction of the negative x axis. Here $v_x = dx/dt$ is equivalent to the golf ball's instantaneous horizontal velocity, and k is a small constant. Show that for $kt << 1$, the horizontal distance travelled is approximately

$$x \approx \frac{ut}{\sqrt{2}} - \frac{kut^2}{2\sqrt{2}}.$$

If $k = g/10u$, show that D is approximately 7% shorter than the case in the absence of the wind.

[7 marks]

13. A particle of mass m moves along the x-axis, attracted toward a fixed point P by a force proportional to the distance away from P. The constant of proportionality is equal to k. Initially the particle is at a distance x_0 from P and is given a velocity v_0 away from P. Determine the particle's position and its velocity at any time.

[9 marks]

Determine the amplitude, period and frequency of the motion, and show that the particle reaches a maximum speed of

$$\sqrt{v_0^2 + kx_0^2/m}.$$

[6 marks]