PAPER CODE NO.
MATH 122

THE UNIVERSITY
of LIVERPOOL

SUMMER 2004 EXAMINATIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Mathematics : Year 1
Master of Mathematics : Year 2
Master of Physics : Year 1

DYNAMIC MODELLING

TIME ALLOWED : **Two Hours and a Half**

INSTRUCTIONS TO CANDIDATES

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.
Take $g = 9.81\text{ms}^{-2}$. Give numerical answers to 3 significant figures.
You may use

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

SECTION A

1. Unmolested the concentration of a species of phytoplankton would grow at a rate of 5% per day, but grazing losses due to a predatory species of zooplankton impose a constraint, decreasing the concentration by a rate equivalent to 3% per day. Write down a differential equation for $C(t)$ the phytoplankton concentration at a time *t*, where the unit of *t* is one day. If the phytoplankton concentration at $t = 0$ was C_0, how many days will it take for the concentration to treble?

[6 marks]

2. A dead body has been discovered in a house, and the police, suspecting foul play, wish to determine the time of death. The temperature $\theta(t)$ of the body at time *t*, is assumed to decrease at a rate proportional to the difference between θ and the ambient room temperature T_R, with constant of proportionality *k*. When the body was first found, at $t = 0$, its temperature was $30^0 C$ and 2 hours later this had fallen to $28^0 C$. Write down a differential equation for $\theta(t)$, and integrate it to show that

$$k = \frac{1}{2} \ln\left[\frac{30 - T_R}{28 - T_R}\right].$$

If $T_R = 20^0 C$ and assuming the body's temperature was the normal $37^0 C$ at the time of death, how long had it lain undiscovered? [9 marks]

3. A house buyer takes out a mortgage of £80000 to buy a new house. The building society lends him the money at an interest rate of 0.48% per month and the customer makes monthly repayments of £500 to pay off the loan. Write down the equation for u_{m+1}, the balance owed at the end of month $(m+1)$, in terms of u_m. If $x_m = u_m - N$, where *N* is the equilibrium solution of this equation, write down a second equation for x_m in terms of x_0. How long does it take the customer to pay off his mortgage and how much does he pay in total?

[8 marks]

4. If *n* the number of events that have occurred at time *t*, follows a Poisson process, then the probability $P(n,t)$ that *n* events have occurred by time *t* is given by

$$P(n,t)=\frac{(\lambda t)^n}{n!}e^{-\lambda t},$$

where λ is a constant. At what time does $P(n,t)$ reach its maximum value? If a motor dealer sells, on average, 3 new cars per day, what (assuming the number of cars sold follows a Poisson process) is the probability that the dealer sells 20 new cars in a working week (6 days)?

[5 marks]

5. A particle travels so that its velocity is given by

$$\boldsymbol{v}=\sin(t)\boldsymbol{i}+\boldsymbol{j}\,te^{-t^2}+\cos(t)\boldsymbol{k}\,.$$

Calculate at any time *t*, a) the particle's speed, b) its acceleration and c) its position, given that at $t=0$ it was situated at $(2,-3,1)$.

[6 marks]

6. A ball of mass *m*, initially at rest, is dropped vertically from a tower 30m high. As it falls the ball is subject to a resistive force equivalent to $0.1mv$ N, where *v* is its velocity at time *t*. Write down Newton's equation of motion and show that

$$v=10g\left(1-e^{-0.1t}\right).$$

Integrate this expression exactly to find the distance travelled at time *t*. Assuming that $g=9.81\text{ms}^{-2}$ show that the ball first hits the ground approximately 2.47 seconds after release. [Hint: $e^x\approx 1+x+x^2/2!$, when $|x|<<1$.]

[9 marks]

7. The equation for the displacement *x* of a forced harmonic oscillator is

$$\ddot{x}+36x=16\sin(2t).$$

At time $t=0$, $x=0$ and $\dot{x}=4$. Find *x* at time *t*. What is the next time $t>0$ when $x=0$ again?

[7 marks]

8. Let $x(t)$ and $y(t)$ represent the levels of two populations governed by the following coupled differential equations

$$\frac{dx}{dt} = (15 - y) \qquad\qquad \frac{dy}{dt} = 5(x - 5).$$

Initially $x = 7$ and $y = 11$. Obtain and solve the differential equation for y in terms of x. From your results draw a phase diagram for this situation, indicating which way around the curve the point (x, y) moves.

[5 marks]

SECTION B

9. Consider a two-state stochastic system, with states A and B. In the usual notation,

$$\frac{d}{dt}P(A,t) = P(B,t)W(B \to A) - P(A,t)W(A \to B).$$

Write down what each term in this equation represents.

[5 marks]

A businessman commutes to work by train. If his train is late on a particular day, the probability that it is late the following day is 0.5. However if his train is on time on a particular day the probability that it is on time the next day is 0.95. If $P(OT,t)$ and $P(L,t)$ are the probabilities of the businessman being on time or late for work at time t, show that

$$\frac{dP(OT,t)}{dt} + 0.55P(OT,t) = 0.5.$$

Solve this equation if $P(OT,0) = 0.8$. The businessman cannot afford to be late more than 5% of the time. Is his probability of being late ever likely to fall below this figure, or should he seek alternative transport?

[10 marks]

10. Let $x(t)$ be the population of lions in a game reserve at time *t* in years and $y(t)$ be the corresponding population of buffalo upon which they prey. Suppose these populations satisfy the differential equations

$$\frac{dx}{dt} = 4(800 - y)$$
$$\frac{dy}{dt} = (x - 200)$$

The game warden carries out an initial survey in which he estimates the population of lions to be 248 and buffalo to be 768. Obtain and solve the differential equation for *y* in terms of *x*. From your results draw a phase diagram for this situation, indicating which way around the curve the point (x, y) moves. [8 marks]

By using the above equations derive the second order equation

$$\frac{d^2x}{dt^2} + 4x = 800 .$$

Solve this equation to find $x(t)$ and hence $y(t)$. Deduce that it takes about 15 months after the survey for the buffalo population to reach its maximum. [7 marks]

11. A sheep farmer in the Australian outback maintains a large flock of sheep whose population $n(t)$, where *t* is the time in years, satisfies the differential equation

$$\frac{dn}{dt} = 8n - \frac{n^2}{40} .$$

Find the equilibrium values of the flock and determine their stability. [5 marks]

Integrate the above equation (use partial fractions) to find $n(t)$ assuming $n = 200$ when $t = 0$. [5 marks]

The farmer wishes to sell 150 of his sheep at the market each year. The proposed sale would alter the differential equation to

$$\frac{dn}{dt} = 8n - \frac{n^2}{40} - 150 .$$

Find the equilibrium population values and their stability in this case. Decide, by means of a diagram, if the farmer can sell 150 animals at market and ensure the long-term size of his flock doesn't fall below 250 animals. (Again assume $n = 200$ when $t = 0$.) [5 marks]

12. A parachutist jumps from a plane and falls freely for 5 seconds before opening his parachute. The mass *m* of the parachutist, together with his clothes and equipment, is 75kg. The drag due to air resistance is proportional to the *square* of his velocity *v*, the proportionality constant $k = 0.3\,\mathrm{kgm}^{-1}$ in free fall and $k = 150\,\mathrm{kgm}^{-1}$ with the parachute open.
Assuming the downward direction is positive, write down Newton's equation of motion linking dv/dt and the forces acting on the parachutist. Integrate this equation (assuming $v = 0$ when $t = 0$) either using the substitution $v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\theta\right)$ or by partial fractions to find *v* in terms of *t*. Taking $v = dy/dt$, integrate your result to show that given $y = 0$ when $t = 0$,

$$y = \frac{m}{k}\ln\left(\cosh\left(\sqrt{\frac{kg}{m}}t\right)\right).$$

[10 marks]

Find how far the parachutist falls before he opens his parachute. Find also the least possible value of his speed as he reaches the ground.

[5 marks]

13. The lookout on a large ship travelling east at 25km/h observes a small (too small to appear on the ship's radar) speedboat some 800m due north of his position. It appears to be travelling south-west, at a speed of 35km/h.
By using vector methods, find the actual velocity and speed of the speedboat.

[6 marks]

Find the position vector at time *t* of the speedboat with respect to the large ship.

[3 marks]

The lookout fears the speedboat will be swamped by the ship's wake if it gets closer than 200m. Does he need to radio to the ship's captain to order an emergency change of course?

[6 marks]