Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

Take $g = 9.81$ m s$^{-2}$. Give numerical answers to 3 significant figures.

You may use

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}.$$

## SECTION A

**1.** An empty cylindrical kettle is placed under a cold water tap. The tap is turned on at $t = 0$ s. For the first two seconds,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.2t$$

where $h(t)$ cm is the height of the water in the kettle at time $t$ s. Solve this equation to find the height at 2 seconds.

After two seconds, the height increases at the constant rate of 0.4 cm $\mathrm{s}^{-1}$ for a further ten seconds. Write down and solve the differential equation for this period.

Then, for a further four seconds, the rate of increase of height is given by $0.1(16 - t)$ cm $\mathrm{s}^{-1}$. Write down and solve the differential equation for this period.

Deduce that the final height of water in the kettle is 5.2 cm.

[7 marks]

**2.** Newton's law of cooling is expressed mathematically as

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_s).$$

What is $T_s$?

By defining $x = T - T_s$, determine the differential equation for $x$. *Write down* the solution of this equation.

A dead turkey at a temperature of $20^o$C is placed in a large commercial freezer maintained at a temperature of $-20^o$C. Using $k = 2 \times 10^{-4}$ $\mathrm{s}^{-1}$, determine how long it takes for the turkey to cool to $-10^o$C.

[7 marks]

**3.** A country road has one lane each way. Repairs are necessary for 100 metres of one lane. Traffic lights are installed so that traffic uses the second lane, alternating in direction. Vehicles pass through a green light at the rate of 20 vehicles per minute. Vehicles travelling north arrive at one set of lights, which are green for $T_1$ minutes, at the rate of 6 per minute. Southbound vehicles arrive at the second set of lights, which are green for $T_2$ minutes, at the rate of 10 per minute. *Both* sets of lights are at red after *each* green light for 1 minute in order for the open lane to clear.

Show that to avoid congestion, $20T_1 \geq 6(T_1 + T_2 + 2)$ and $T_2 - T_1 \geq 2$. Shade the region in the $T_1 - T_2$ plane for the times satisfied by these inequalities. Deduce that $T_1$ has to be at least 3 minutes and find the minimum value of $T_2$.

[8 marks]

**4.** This year there was an advertisement in a magazine offering Personal Loans of £5000 at a rate of 8.9% per annum. Show that this is equivalent to a rate of 0.713% *per month.* Write down the equation for $u_{m+1}$, the balance owed at the end of month $m + 1$ in term of $u_m$ and $c$, the fixed amount repaid each month. Let $N$ be the equilibrium solution of this equation and $p_m = u_m - N$. *Write down* the equation for $p_m$ in terms of $p_0$. Given $u_{60} = 0$, show that

$$N - 5000 = (1.089)^{-5} N .$$

Solve this equation for $N$ and hence, or otherwise, calculate $c$ and the total amount paid during the 5 years for the loan.

[9 marks]

**5.** At time $t$, the position vector of a particle is given by

$$\mathbf{r}(t) = a\cos(\omega t)\ \mathbf{i} - a\sin(\omega t)\ \mathbf{j} + bt\ \mathbf{k}\ ,$$

where $a$, $b$ and $\omega$ are positive constants. What is the name of the path taken by this particle? Find its velocity, $\mathbf{v}(t)$, and its acceleration at time $t$. Show that

$$\frac{d^2\mathbf{r}}{dt^2} = \omega \mathbf{v} \times \mathbf{k}.$$

[7 marks]

**6.** The engines of a ship, travelling in a straight horizontal line with the constant speed 4 m $s^{-1}$, stop at $t = 0$. The ship (of mass $m$ kg) then slows through a non-constant resistive force so that its Newton's equation of motion is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{200}mv^{\frac{1}{2}} \quad \mathrm{m\ s^{-2}}.$$

Show that the ship comes to rest after a time 800 s. Find the distance travelled by the ship during this time.

[6 marks]

**7.** A particle of unit mass moves on the $x-$axis with simple harmonic motion about the origin. Its amplitude is 2 cm and its period is $\pi$ seconds. Use the equation of motion and the definition of velocity,

$$\frac{dv}{dt} = -\omega^2 x, \qquad \frac{dx}{dt} = v \ ,$$

to show that

$$v^2 + 4x^2 = 16 \ .$$

Sketch the phase diagram $v$ versus $x$. Show with the use of arrows on your phase diagram the direction which shows the particle's motion. [6 marks]

**8.** On average, 3 e-mails per hour arrive at my PC. Assuming that the arrival of e-mails is a Poisson process, find the probability that exactly 10 e-mails arrive between 09:00 and 13:00.

[5 marks]

## SECTION B

**9.** Consider a two-state stochastic system, with states $A$ and $B$. In the usual notation,

$$P(A, t + \delta t) = P(B, t)W(B \to A)\delta t + P(A, t)(1 - W(A \to B)\delta t) \ .$$

Briefly explain what this equation represents.

Derive an equation for the rate of change of $P(A, t)$. [5 marks]

A disease infects people in such a way that if they feel ill on one day, the probability that they feel ill on the next is 0.7. If however they feel well on one day the probability that they feel well the next day is 0.6. Consider this as a two-state process and write down the equation for the rate of change of $P(well, t)$ in terms of $P(well, t)$ and $P(ill, t)$ the probabilities respectively of feeling well and ill on day $t$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}P(well, t) = 0.3 - 0.7P(well, t) \ .$$

Solve this equation to find $P(well, t)$, given that $P(well, 0) = 1.0$, and find the long-term value of $P(well, t)$. [10 marks]

**10.** Let $x(t)$ represent the number of hares and $y(t)$ the number of foxes in a particular population at time $t$ weeks. Suppose that these satisfy the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{4}(300 - y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{9}(x - 400) \ .$$

Initially there are 300 foxes and 100 hares.

Obtain and solve the differential equation for $y$ in terms of $x$.

From your results draw a phase diagram for this situation, indicating which way around the curve the point $(x, y)$ moves. [8 marks]

By using the above equations derive the second order equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{1}{36}x = \frac{400}{36} \ .$$

Solve this equation to find $x(t)$ and hence $y(t)$.

Deduce that it takes about 38 weeks for there again to be 300 foxes and 100 hares.

[7 marks]

**11.** Show that if $n = N$ is an equilibrium value of the differential equation

$$\frac{dn}{dt} = f(n) ,$$

then the equilibrium is stable if $f'(N) < 0$, where prime denotes differentiation with respect to $n$. [5 marks]

In the absence of fishing, the number, $n(t)$ million, of fish in a particular sea satisfies the differential equation ($n > 0$),

$$\frac{dn}{dt} = 5n - n^2 \quad \text{per year.}$$

Fishing quotas are introduced which allow only four million fish to be caught each year. What are the equilibrium values of $n$ for this quota? (Any other external effects such as bad weather, disease, etc are ignored.) Which of these values are stable?

In each of the following cases, determine the long term behaviour of $n(t)$ when fishing takes place and the initial number of fish are (i) 500,000, (ii) 3,000,000 and (iii) 5,000,000.

[7 marks]

Suppose that the fishing quota is increased. What is the critical number of fish that can be caught each year to prevent overfishing, i.e. the fish becoming extinct? [3 marks]

**12.** A golf ball of mass $m$ is hit from ground level on a horizontal golf course with a speed $u$ at an angle of $\frac{1}{4}\pi$ above the horizontal. The only force acting on the ball during its flight is that due to gravity.

Write down the two second order differential equations for the horizontal and vertical distances, $x$ and $y$ travelled by the ball in $t$ seconds. Solve these equations to find $x$ and $y$ as functions of $t$.

Find, in terms of $u$ and $g$, the time taken to the first bounce and show that this is at a distance of

$$R = \frac{u^2}{g} .$$

[7 marks]

On the next day, a similar golf ball is hit with the same velocity. However, in addition to the force due to gravity, it experiences a constant force of $\frac{1}{10}mg$ due to a wind blowing in the direction of the positive $x-$axis.

Show that, compared to the first ball, the time of flight is the same but the distance to the first bounce is 10% greater.

Show that $\dot{y} + 10\dot{x}$ is a constant of the motion. [8 marks]

**13(a).** A particle of mass $m$ is thrown vertically upwards with an initial velocity $u$. It experiences an air resistance of $mkv$, when its velocity is $v$, where $k$ is a constant.

Show that it reaches a height

$$h = \frac{u}{k} - \frac{g}{k^2} \ln\left(1 + \frac{ku}{g}\right).$$

[7 marks]

**(b).** A bungee jumper of mass 70 kg, initially at rest, drops from a bridge 60 m above a river and then falls under the force of gravity. The unstretched rope has length 20 m.

For $y \geq 20$, the rope exerts a force $-14g(y - 20)$ N, where $y$ is the distance downwards from the bridge. Show that the energy stored in the taut rope is

$$7g(y - 20)^2 \text{ J}.$$

What is the maximum distance that she falls? [8 marks]