Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

Take $g = 9.81$ m s^{-2}. Give numerical answers to 3 significant figures.

You may use

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}.$$

SECTION A

1. On a certain day, 100 students arrive and 50 students leave the university campus per minute. Write down the differential equation for $n(t)$, the number of students on campus at time t.

Solve this equation given that there are 1250 students on campus at 9:00am.

For another university, the overall rate of increase of students per minute is given by $\frac{1}{4}(240 - t)$. Write down the differential equation for $n(t)$ for this case and solve it using the same initial conditions as before. Show that this campus has no students at 17:20. [7 marks]

2. The temperature, T^oC, of a cup of coffee in the staff room at time t satisfies the differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} + kT = 20k,$$

where k is a constant. If the temperature of the coffee is initially 80^oC, solve this equation to find the temperature at time t. Use your result and Newton's Law of Cooling to find the temperature of the staff room.

If the coffee cools to 50^oC in 5 minutes, find the value of the constant k to 3 significant figures. [6 marks]

3. A mosquito population increases at the rate of 10% per hour. One thousand mosquitos per hour are killed as the result of an extermination programme. Given that at the start of this programme there are 9000 mosquitos and that it takes T hours to kill all the mosquitos, show that T satisfies

$$T = 10\int_0^{9000} \frac{\mathrm{d}n}{10000 - n}.$$

Evaluate this integral to show that the exercise is completed within 1 day.

[6 marks]

4. At the end of the m'th year, a borrower owes £u_m to a Building Society for the mortgage on his house, where u_m satisfies

$$u_{m+1} = 1.08u_m - 4000 \text{ pounds.}$$

What is the interest rate charged by the Building Society and how much does he repay each year?

What is the equilibrium solution, N, of this equation?

Show that $p_m = (1.08)^m p_0$, where $p_m = u_m - N$.

Given that $p_0 = -£9933$ verify that the mortgage will be fully paid after 21 years. Find the initial value of the loan and the total amount of the repayment.

[9 marks]

5. Initially a particle is at the point (1,1,0). At time t, its velocity is given by

$$\mathbf{v} = 2\mathbf{i} - 4t\mathbf{j} + (5-t)\mathbf{k} \text{ m s}^{-1} .$$

Find its speed at time $t = 1$ second.

Find (i) its acceleration and (ii) its position vector at time t.

[7 marks]

6. A particle of mass m moves horizontally on a rough surface in the positive $x-$direction . It experiences the non-constant resistive force mkv^p, where k and p are positive constants.

Write down Newton's equation of motion.

Show that the particle will never come to rest if $p > 1$.

Show also that if $p = 1/2$ and the particle's speed as it passes through the origin is u, the particle will come to rest at the point $x = 2u^{3/2}/3k$.

[8 marks]

7. Newton's equation of motion for a particle moving under dampened harmonic motion is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -m\omega^2 x - mk^2\frac{\mathrm{d}x}{\mathrm{d}t}$$

where ω and k are positive constants.

By multiplying this equation by $\frac{\mathrm{d}x}{\mathrm{d}t}$, show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2\right] = -mk^2\dot{x}^2.$$

Deduce that the energy $E = T + V$ (in the usual notation) is not conserved and decreases with time. [6 marks]

8. On average, 10 cars per hour pass a particular house. Assuming that this is a Poisson process, find the probability that exactly 40 cars pass the house between the hours 09:00 and 13:00. [6 marks]

SECTION B

9. Explain what is meant by a stochastic process. [2 marks]

Consider a two-state stochastic system. At time t, let $P(A,t)$ be the probability that the system is in the state A and $P(B,t)$ the probability that it is in the state B. Let $W(B \to A)$ be the probability per unit time the system goes from state B to state A and let $W(A \to B)$ be the probability per unit time the system goes from state A to state B.

Show that the rate of change of $P(A,t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}P(A,t) = P(B,t)W(B \to A) - P(A,t)W(A \to B)$$

[5 marks]

If it is raining one day the probability of rain the next day is $\frac{1}{4}$. If it is fine on one day, the probability of rain the next day is $\frac{1}{2}$.

Write down and solve the differential equation for $P(\text{rain},t)$, given that the weather is fine when $t = 0$.

Show that $P(\text{rain},t) \to \frac{2}{5}$ as $t \to \infty$.

[8 marks]

10. In a simplified model for an animal population Y, with $y(t) \times 10^3$ animals at time t, which is the principal predator for another animal population X, with $x(t) \times 10^3$ animals at time t, the differential equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2-y}{5}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{x-1}{5}. \qquad (x > 0,\ y > 0).$$

Initially there are 2000 animals of type X and 1000 of type Y.

Obtain and solve the differential equation for y in terms of x. [6 marks]

From your results draw a phase diagram for this situation. How many Y animals are left when the X population is extinct? [9 marks]

11. A gnat population increases at the rate of $(n^2 + 4) \times 10^4$ per hour, where $n(t) \times 10^4$ is the number of gnats at time t. For a particular eradication programme they are killed at the rate of $5n \times 10^4$ per hour.

Write down the differential equation for n.

Find the equilibrium values of n and, with the aid of a diagram, discuss the stability for each one.

Show, by using the method of partial fractions or otherwise, that

$$\frac{n-4}{n-1} = A\mathrm{e}^{3t},$$

where A is a constant.

Given that there are 2×10^4 gnats at time $t = 0$, show that

$$n = 1 + \frac{3}{1 + 2\mathrm{e}^{3t}}.$$

[15 marks]

12. A tennis ball is hit from a height of 2m on a horizontal tennis court with a speed of $\sqrt{82g}$ ms^{-1} at an angle of $\arctan(\frac{1}{9})$ above the horizontal.

Write down the two second order differential equations for the horizontal and vertical distances, x and y travelled by the ball in t seconds. Solve these equations to find x and y as functions of t. Use these solutions to show that the path taken by the ball is

$$y = \frac{5}{2} - \frac{1}{162}(x-9)^2.$$

Find the maximum height reached by the ball and the horizontal distance travelled before its first bounce.

[15 marks]

13. One end of a light elastic *string* of length L and modulus $\lambda = 12mg$ is attached to a fixed point O on a ceiling. A particle of mass m is attached to the other end of the string and is initially held at O.

At time $t = 0$ the particle is released and begins to fall. The $y-$axis is vertically downwards. Air resistance is neglected. Find the time t_1 before the string first becomes taut.

For $t_1 < t < t_2$, where t_2 is the time when the string next becomes slack, write down expressions in terms of y and $\dot{y}$ for (i) the kinetic energy, (ii) the gravitational potential energy and (iii) the energy stored in the string.

Find the maximum value of y. [11 marks]

When the particle reaches its lowest point a stop watch is started. At time τ on the stop watch and while the string is stretched, y is given by

$$y = \frac{13}{12}L + \frac{5L}{12}\cos(\omega\tau).$$

where $\omega = \sqrt{12g/L}$. Show that the total time for the particle to perform one cycle is

$$\frac{2}{\omega}\left[\sqrt{24} + \arccos\left(-\frac{1}{5}\right)\right].$$

[4 marks]