Instructions to candidates

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

Take $g = 9.81$ m s^{-2}. Give numerical answers to 3 significant figures.

You may use

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}.$$

SECTION A

1. After the end of a 'pop' concert fans there are 60,000 fans in the arena. Given that they leave the arena at the rate of 12000 per minute, write down the differential equation for $n(t)$, the number of fans in the arena at time t.

Solve this equation to find how long it takes to empty the arena .

In an identical concert, with the same number of fans, the fans leave the arena at the rate of $1200(10 - t)$ per minute. Write down the differential equation for $n(t)$ for this case and find how long it takes to empty the arena [6 marks]

2. State Newton's law of cooling. A turkey is taken out of a freezer at -15^oC. It is left in a room in which the temperature is 20^oC. Show that if Newton's constant is 1.45×10^{-5} s^{-1} it takes about 1 day for the turkey to reach a temperature of 10^oC.

[You may assume that Newton's law also applies to warming.] [7 marks]

3. Consider two one-way roads which cross at right angles. There are traffic lights at the intersection. Cars arrive at the rate of 4 per minute at one set of lights which are at green for a time T_1 mins. At the other set, cars arrive at the rate of 3 per minute and the green light lasts for T_2 mins. In addition both lights are red for one minute at the same time to allow pedestrians to cross. Traffic passes a green light at the rate of 10 cars per minute.

Show that to avoid congestion, $10T_1 \geq 4(T_1 + T_2 + 1)$ and $6T_1 - 4T_2 \geq 4$. Shade the region in the $T_1 - T_2$ plane for the times satisfied by these inequalities. Deduce that T_2 has to be greater than 1 minute. [7 marks]

4. Bill's bank balance, $n(m)$, at the end of year m is given by

$$n(m) = 0.95n(m - 1) + 1000 \text{ pounds},$$

where $n(m - 1)$ is the balance at the end of the previous year.

What is the equilibrium solution, N, of this equation? Given that $p(m) = n(m) - N$, show that

$$p(m) = (0.95)^m p(0).$$

If the balance at the end of year 0 is £1000, what is the balance, to the nearest penny, at the end of year 15? [6 marks]

5. At time t, a point on a fairground ride has acceleration

$$\frac{d\mathbf{v}}{dt} = 5\cos\left(\frac{1}{2}t\right)\mathbf{i} + 5\sqrt{2}\sin\left(\frac{1}{2}t\right)\mathbf{j} - 5\cos\left(\frac{1}{2}t\right)\mathbf{k} \text{ m s}^{-2}.$$

At time $t = 0$ s, the point passes through $-20\mathbf{i} + 60\mathbf{k}$ with a velocity $-10\sqrt{2}\mathbf{j}$ m s^{-1}. Find its position vector of this point at time t.

Show that the distance form the point $40\mathbf{k}$ m is always $20\sqrt{2}$. [7 marks]

6. The engines of a ship stop at $t = 0$. The ship (of mass m kg), travelling in a straight line, slows through the resistive force $mkv^{\frac{1}{2}}$ N. Write down Newton's equation of motion.

Given that the initial speed is u m s^{-1}, find the time taken (T) and the distance travelled (D) before the ship comes to rest.

Deduce that $uT = 3D$. [8 marks]

7. A particle experiences simple harmonic motion with amplitude a about the origin. Use Newton's equation of motion,

$$\ddot{x} + \omega^2 x = 0,$$

to show that its speed v satisfies

$$v = \omega(a^2 - x^2)^{\frac{1}{2}}.$$

Such a particle passes through the origin with a speed 10 cm s^{-1} and the point $x = 4$ cm with a speed 6 cm s^{-1}. What is the period for this motion? [7 marks]

8. A golf ball is hit from the tee at the origin with a speed of u m s^{-1} at an angle of 60° to the horizontal. Show that the equation of the path taken by the ball is given by

$$y = \sqrt{3}x - 2\frac{g}{u^2}x^2.$$

Find the value of u if the first bounce of the golf ball is 100 metres from the tee. [7 marks]

SECTION B

9(a). The speed v ms^{-1} of a snowplough when the depth of the snow is z m satisfies the differential equation $\frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{3}{2}z^2$. If the speed of the snowplough is 4 ms^{-1} when there is no snow, what is the minimum depth of the snow that prevents the snowpough from working? [4 marks]

(b). On average, 3 letters per delivery arrive at my house. Assuming that the incidence of delivery is a Poisson process, find the probability that exactly 12 letters arrive in 6 deliveries? [4 marks]

(c). Each day a student uses a PC to access the Internet. Sometimes she gains immediate access and at other times she has to wait for a while. If she has to wait one day the probability that she does not have to wait the next day is 0.6 and vice-versa. Show that the probability that she has to wait on day t, $P(wait, t)$, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}P(wait, t) + 1.2P(wait, t) = 0.6.$$

Given that $P(wait, 0) = 0$, solve this equation and find the probability that she does not have to wait on day 4. [7 marks]

10. Let $x(t)$ represent the number of rabbits and $y(t)$ the number of foxes in a particular population at time t months. In a simplified model, these satisfy the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 90 - y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - 40 \qquad (x > 0,\ y > 0).$$

Initially there are 90 foxes and 70 rabbits.

(i) Obtain and solve the differential equation for y in terms of x. Deduce that the point (x, y) lies on a circle of radius 30. Sketch this circle and show the direction in which this point moves. [7 marks]

(ii) Obtain a second order differential equation for y in terms of t. Solve this equation and hence find x and y as functions of t. [8 marks]

11. The number $n(t)$ of oysters in a particular oyster bed satisfy the differential equation ($n > 0$),

$$\frac{\mathrm{d}n}{\mathrm{d}t} = 24n - \frac{n^2}{500} - f \quad \text{per year}$$

where f is the effect of fishing.

If there is no fishing, what is the equilibrium number of oysters? [3 marks]

However, there is fishing. Given that $f = 40,000$ oysters per year, sketch the graph of $\mathrm{d}n/\mathrm{d}t$ against n and on this graph indicate how $n(t)$ behaves for various values of $n(0)$. Deduce that this amount of fishing is acceptable. [10 marks]

Give advice, with supporting mathematical evidence, on the wisdom of increasing the fishing to 72,000 oysters per year. [2 marks]

12. A stone of mass m is dropped vertically down a well. In addition to gravity it experiences a resistive force mkv, where k is a constant and v is its speed at time t after it has travelled a distance y (take the $y-$ axis vertically downwards). Write down Newton's equation of motion. Solve this equation and show that it takes a time

$$T = -\frac{1}{k}\ln\left(1 - \frac{ku}{g}\right)$$

to reach the speed u. [7 marks]

Express Newton's equation of motion in terms of a derivative with respect to y. Given that when $v = u$, the distance travelled is D, show that

$$D = \frac{1}{k}\int_0^u \left[\frac{g}{g - kv} - 1\right]\mathrm{d}v.$$

Hence, or otherwise, show that

$$u = gT - kD.$$

[8 marks]

13. A light elastic *string* of length L has modulus $\lambda = 4mg$. The string is stretched to a length y. Show that the energy stored in the string is

$$2mg\frac{(y - L)^2}{L}.$$

[3 marks]

One end of the string is attached to a fixed point O on a ceiling. A particle of mass m is attached to the other end of the string and the system is suspended vertically from O. The $y-$axis is vertically downwards. What is the value of y when the system is at rest? [3 marks]

The particle is pulled down to $y = y_{max}$ and then released from rest. Find the value of y_{max} for which the particle will subsequently just touch O. [9 marks]