MATH104 Exam May 2007, Solutions

All questions except q.1 are standard homework examples

1. tau, omega, θ, ρ. (1 mark each.)

2.

a) $f(n) = -18 + 10n$.

b) $f(n) = 4(n + 1)$

c) $f(n) = n$ if n is even and $-n - 1$ if n is odd.

(2 marks for a), 4 marks for b) and 6 marks for c).)

3.

a) $a \neq 1$ and $a < 2$. (2 marks)

b) $x \leq y$ or $y \leq z$ (3 marks)

c) There exist x and y with $f(x) < f(y)$ and $x \geq y$ (3 marks)

d) For all $N \in \mathbb{N}$ there exists $x \in \mathbb{R}$ such that $f(x) \geq N$ or $g(x) \geq N$. (4 marks)

4.

a) f is *not injective* if there exist $x, y \in \mathbb{R}$ such that $f(x) = f(y)$ but $x \neq y$. (2 marks)

b) Let $x, y \in \mathbb{R}$ and suppose $f(x) = f(y)$. Then $10-3x = 10-3y$ and rearranging gives $3x = 3y$ so $x = y$. (3 marks)

c) Let $x = 0, y = -1$. Then $f(x) = f(y) = 0$ but $x \neq y$. Hence f is not injective. (4 marks)

d) Let $x, y \in \mathbb{R}, x \geq 0, y \geq 0$, Suppose that $f(x) = f(y)$. Then $x^2 + x = y^2 + y$, which gives $x^2 - y^2 + x - y = 0$, that is $(x - y)(x + y) + (x - y) = 0$, that is $(x - y)(x + y + 1) = 0$. But $x > 0$ and $y > 0$ so that the second factor cannot be 0. Hence the first factor is 0, giving $x = y$. (4 marks)

5.

For integers m and n the statement $m|n$ means that there exists an integer k with $n = km$. (1 mark)

a) R is not an equivalence relation. Property (ii) fails, since for example $1\,R\,2$ is true but $2\,R\,1$ is false (since $2 = 1 + 1$). (3 marks)

b) R is an equivalence relation. For let x, y, and z be any integers. Then

i) $x - x = 0$ and $10|0$ since $0 = 0 \times 10$, so $x\,R\,x$.

ii) If $x\,R\,y$ then $x - y = 10k$, for some $k \in \mathbb{Z}$ so $y - x = 10(-k)$, and $-k \in \mathbb{Z}$ so $y\,R\,x$.

iii) If $x\,R\,y$ and $y\,R\,z$ then $x - y = 10k$ and $y - z = 10l$ for some integers k, l so $x - z = (x - y) + (y - z) = 10(k + l)$, i.e. $x\,R\,z$.

(4 marks)

c) R is not an equivalence relation. In this case only property (iii) fails. For example, $2\,R\,6$ since 2 divides 2 and 6, and $6\,R\,9$ since 3 divides 6 and 9, but $2\,R\,9$ is false since there is no prime dividing both 2 and 9. (6 marks)

6.

a) Let m and n be integers and suppose that m and n are both odd. Then $m = 2k+1,\ n = 2l+1$ for some integers $k,\ l$. Hence $mn = (2k+1)(2l+1) = 2(2kl+k+l)+1$, which is of the form $2k'+1$ for an integer k' and hence odd.

The converse states: Let $m, n \in \mathbb{Z}$. If mn is odd then m and n are odd. This is *true*: Suppose that mn is odd and that m or n is even, say m. Then $m = 2k$ for an integer k so that $mn = 2kn$ is even, giving a contradiction. [Or we could prove the contrapositive: if m is even or n is even, then mn is even, in the same way.] (6 marks)

b) Suppose, for a contradiction, that m and n are integers and that $6m + 9n = 22$. Then $3(2m + 3n) = 22$ and it follows that $3|22$. But this is false since $22 = 3 \times 7 + 1$. This contradiction proves the result. (3 marks)

c) The contrapositive is:
Let $a, b \in \mathbb{R}$. If $(a + b)^2 \leq 4ab$ then $a = b$.

Proof: From $(a + b)^2 \leq 4ab$ we have $a^2 + 2ab + b^2 \leq 4ab$, so $(a - b)^2 \leq 0$. This is possible only if $a - b = 0$, that is $a = b$. (6 marks)

7.

Context X is a subset of euclidean n-space. (1 mark)

Hypothesis X is closed and X is bounded. (1 mark)

Conclusion X is compact.(1 mark)

Contrapositive *Let X be a subset of euclidean n-space. If X is not compact then X is not closed or X is not bounded.* (2 marks)

a) Nothing: hypothesis of theorem not fully satisfied. (2 marks)

b) X is not closed or X is not bounded. From contrapositive.(2 marks)

c) X is not bounded. From (b) and ruling out 'not closed'. (2 marks)

d) Nothing: not hypothesis of theorem or contrapositive. (2 marks)

However if the converse of the theorem is true then X compact would imply *both* X closed *and* X bounded. Since neither is the case we can deduce X is *not* compact. (We could deduce this merely from *one* of 'X not closed', 'X not bounded'.) (2 marks)

8.

a) False. ($x = 0$ gives $\cos x = 1$.) (2 marks)

b) True. (Take $n = -3$.) (3 marks)

c) False. (Take $x = 2$. Then there is no y with $\cos y = x$.) (3 marks)

d) True. (Take $m = 1$.) (3 marks)

e) False. (Take $y = \frac{1}{2}$. Then $x < y - 1 = -\frac{1}{2}$ cannot hold for $x \in \mathbb{R}^+$.) (4 marks.)