THE UNIVERSITY
of LIVERPOOL

1. Give the names of the following (lower case) Greek letters: τ, ω. Write the lower case Greek letters *theta* and *rho*. [4 marks]

2.

For each of the following sets S, give a function $f(n)$ such that

$$S = \{f(n) \mid n \in \mathbb{N}\}.$$

(Recall that $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.)

a) $S = \{-18, -8, 2, 12, 22, 32, \ldots\}$.

b) $S = \{m \in \mathbb{N} \mid m > 0 \text{ and } m \text{ is a multiple of } 4\}$.

c) S is the set of all even integers, that is $S = \{\ldots -6, -4, -2, 0, 2, 4, 6, \ldots\}$.

[12 marks]

3. Negate each of the following statements:

a) $a = 1$ or $a \geq 2$.

b) $x > y > z$.

c) If $f(x) < f(y)$ then $x < y$.

d) $\exists N \in \mathbb{N}, \forall x \in \mathbb{R}, f(x) < N$ and $g(x) < N$. [12 marks]

4.

Definition: Let $f(x)$ be a (real-valued) function. Then $f(x)$ is *injective* if for all x, y in the domain of f,

$$f(x) = f(y) \implies x = y.$$

a) Write down what it means for f *not* to be injective.

b) Prove that the function $f(x) = 10 - 3x$, $x \in \mathbb{R}$ is injective.

c) Prove that the function $f(x) = x^2 + x$, $x \geq -1$ is not injective.

d) Prove that the function $f(x) = x^2 + x$, $x \geq 0$ is injective.

[13 marks]

5. Write down carefully the meaning of the statement that $m|n$ ('m divides n'), where m and n are integers.

Definition: Let R be a relation on a set X. Then R is an *equivalence relation* if for all $x, y, z \in X$ the following three conditions hold:

i) $x\,R\,x$.

ii) If $x\,R\,y$ then $y\,R\,x$.

iii) If $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$.

Determine whether or not the following relations R on the given sets X are equivalence relations. You should justify your answers carefully, working directly from the definitions.

a) $X = \mathbb{Z},\ x\,R\,y$ if $x \neq y + 1$.

b) $X = \mathbb{Z},\ x\,R\,y$ if $10|(x - y)$.

c) $X = \{n \in \mathbb{N} \,|\, n \geq 2\},\ x\,R\,y$ if there is a prime number which divides both x and y. (Recall that a prime number is an element of the same set X which has no factor other than itself and 1.) [14 marks]

6.

a) Prove the following proposition.
Let m and n be integers. If m and n are odd then mn is odd.

State the converse of this proposition and also state, with a proof, whether the converse is true.

b) Prove the following proposition:
There do not exist integers m and n such that $6m + 9n = 22$.

c) Let P be the following proposition:
Let $a, b \in \mathbb{R}$. If $a \neq b$ then $(a + b)^2 > 4ab$.
State and prove the contrapositive of P. [15 marks]

7. Consider the following theorem. You are not expected to understand what it means.

Theorem *Let X be a subset of euclidean n-space. If X is closed and X is bounded then X is compact.*

Identify the context, hypothesis and conclusion of this theorem. State its contrapositive.

What, if anything, can you deduce from this theorem, given a subset X of euclidean n-space which is

a) closed?

b) not compact?

c) closed and not compact?

d) neither closed nor bounded?

Give a brief reason in each case. Would your answer to (d) change if you are also told that the converse of the above theorem is true? [15 marks]

8. Determine whether each of the following statements is true or false, giving a brief reason in each case. Recall that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
In part e), $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.

a) $\forall x \in \mathbb{R}$, $\cos x < 1$.

b) $\exists n \in \mathbb{Z}$, $n < 0$ and $(n+1)^2 = 4$.

c) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $\cos y = x$.

d) $\exists m \in \mathbb{N}$, $\forall n \in \mathbb{N}$, $mn = n$.

e) $\forall y \in \mathbb{R}^+$, $\exists x \in \mathbb{R}^+$, $x < y - 1$. [15 marks]